A

NEW TREATISE

ON

MECHANICS.

BY THE AUTHOR OF

"A NEW INTRODUCTION TO THE MATHEMATICS,"

"A NEW SUPPLEMENT TO EUCLID'S ELEMENTS OF GEOMETRY,"

&c. &c.

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PREFACE.

The present work has originated in consequence of the extreme brevity of expression, and deficiency of explanation, in the treatises on this subject now in use. If it be very important to maintain conciseness in these works, it should, at least, be employed in stating all that is desirable to be known, and requisite to be explained, on a subject, not of itself difficult to be understood, if the whole of it were clearly laid before the reader, nor indeed difficult to communicate; although from the practice of these authors, it would appear to be impossible. But they are sparing, not only of their words, but of their matter. Nay, in order to avoid wasting language, they will sometimes use a word, not in its ordinary sense (in which they use it themselves in other parts of their works), but in a new meaning, unknown to the beginner; and in this manner they sometimes communicate the most important truths.

For instance, Mr. Bridge, in treating of uniform motion in the very beginning of his work (*Mechanics*, p. 7.) states, that if the space described by a body moving uniformly be *given*, the time of its motion will be in the inverse ratio of its velocity. Now, the ordinary mathematical sense of the

word "given" is "*known;*" and it seems to be made a condition in this proposition, that provided the space described is a *known* quantity, the time will be inversely as the velocity; from which the beginner would infer (what certainly is not meant), that if the space were an *unknown* quantity, the time would *not* be inversely as the velocity, and that their ratio changes, as soon as the space is known or found, from what it was before, while the space was unknown, although the space continues the same unvarying quantity, whether it be known or unknown.

But this short expression saved that author the circumlocution of stating the proposition in words at length, as follows:—"If a body moving uniformly at one time with a certain velocity, and at another time with a different velocity, describes equal spaces, the time in which the space will be described with the greater velocity will be in the inverse ratio of the velocity: the greater velocity will require less time, and the less velocity will require more time." As this truth or proposition is demonstrable from the known properties of uniform motion, it would not have been an unprofitable instruction to have added the demonstration. The reader will observe how pregnant of meaning is the word "given," as used by that author, and he will judge whether a beginner would be likely to ascribe the author's meaning to it, or to take it in its ordinary sense.

The present author, being convinced of the perplexity and mischief occasioned to the student by these sacrifices of important instruction to an overweening and meretricious leaning to brevity, has undertaken the present work, in which he has endeavoured to supply the explanation wanting. The work is, therefore, strictly elementary, and intended for beginners. It embraces the prime principles of Motion, Moving Force, and the Mechanical Powers, without attempting the higher branches usually

treated of by authors on this subject. Indeed, the necessity for such full explanation does not exist as to the higher branches, because the student, when he has acquired not merely the technical rules, but the *rationale* of the elementary parts, will not require much assistance in pursuing his studies further; which, for the most part, will consist merely of the application of the principles which he has previously acquired to other investigations.

Chapter i. part i. contains the Definition and Laws of Motion. The second law of motion is stated by most authors, as follows:—" Motion, or the change of motion, is proportional to the *force impressed*, and is produced in the right line in which that force acts." (*Bridge's Mechanics*, p. 12.) In the present work it is stated thus:—" Motion, or the change of motion, is produced uniformly in the line of direction in which the impulse or force acts, and is proportional to the excess of the force applied above the resistance;" which is materially different from the other. For it appeared to the author, that the reaction of the resistance destroyed an equal quantum of the action of the force originally applied. Thus, if the force impressed were equal to a weight of 2*lbs.*, and if the resistance of the body were equal to a weight of 1*lb.*, the remaining force, $2 - 1 = 1$*lb.*, would be that which moves the body. But let the force be doubled $= 4$*lbs.*, the resistance being 1, the remaining force $4 - 1 = 3$*lbs.* would be that which moves the body; that is, this second motion would be to the first as 3 to 1, and not as 2 to 1, which is the ratio of the forces impressed. This is demonstrated in the chapter on the lever in the second part of this work. (Part ii. chap. i. § 2. art. 8 to 13.)

The third law of motion is stated in chap. i. as follows:— " When a force applied to a body is resisted, the resistance re-acts upon the body in a direction opposite to that of the

force applied, and destroys, *pro tanto*, the action of the force applied." For the usual form of expressing it, "Action and reaction are equal and in opposite directions," is not sufficiently definite, and gives the learner no distinct idea, or perhaps an erroneous one, of what is meant by it. The author has given his own construction of its meaning, which he does not find very distinctly stated in other authors, although they seem to think that the usual form of expression is too comprehensive, and requires some qualification. (*Bridge's Mechanics*, 19 and 20. *Whewell's Mechanics*, art. 187, p. 248.)

In chap. ii. part i. the author has investigated the properties of the Uniform Motion of one body; and in chap. iii. the properties comparatively of the motion of two bodies moving uniformly with different velocities: for these properties may be considered separately with less perplexity, than if they were taken together. Mr. Bridge disposes of both these heads together in less than two pages. In handling these, as well as the other subjects of the work that will admit of it, the author lays down such definitions or truths established by experience and observation as may serve for the foundation of the propositions which he proceeds to advance and to demonstrate. From these demonstrations he deduces the formulæ, by which, from the requisite data, any of the quantities that are sought may be found; and these formulæ are collected in a table in each chapter for ready reference. To which are added problems, in which the use of the different formulæ is manifested.

In chap. iv. part i. the author has, in like manner, treated of the Gravitation of Bodies near the surface of the earth; in chap. v. Gravitation augmented by perpendicular impulse; and in chap. vi. Gravitation counteracted by vertical impulse; because the subject naturally leads to

this division, and becomes less perplexing when each part is taken separately; and their respective formulæ are collected in separate tables.

In chap. vii. and viii. part i. the author has treated on Momentum in general, and the comparative momenta of two bodies moving with uniform velocities; and in treating this branch of the subject he has taken a medium course between those authors who affirm too largely that momentum is *equal* to the product of the weight into the velocity, and those authors who assert barely that the momentum *varies as, or is proportional to,* the product of the weight into the velocity. The author thinks it probable that the motion of a body may be so slow that the momentum generated by it may be less than the weight of the body; for since the body when at rest possesses no momentum whatever, *some* degree of velocity would generate a momentum equal to its weight, and a less velocity than this might be found, which, because the momenta are as the velocities, would generate a momentum less than the weight of the body. The author has therefore taken *d* as the symbol of that quantity of momentum which is equal to the weight of the body; whereby he has obtained an *expression* of the *value* of the momentum. The author has not ascertained the actual value of *d*, but he has suggested experiments for more practised hands to determine it; which, when accomplished, the theory may be easily extended to the same or analogous cases, circumstances, and results, with what has been done with respect to gravitation.

Chap. ix. part i. treats of Moving Force, which may perhaps be considered as that state of a power of which momentum is the transformed state, or that state into which moving force passes, when it is communicated to, and absorbed within, the moving body. The author has defined moving force to be the excess of the force applied above the resistance; and his investigations of the

properties of moving force tend to support the accuracy of this definition. But he must admit that other authors do not always take the same view of it. Mr. Bridge, indeed, (book ii. p. 30.) says, "The force which generates momentum in a body is called the moving force;" and (p. 31.) "the moving force is proportional to the difference of the weights" (over a fixed pulley), which seems confirmatory of this doctrine; but Mr. Whewell seems to entertain a different opinion. "When pressure," says that author, "communicates motion to a body, the moving force is as the pressure." (*Whewell's Mechanics*, art. 180, p. 239.) But, as we have stated above, in the case of the weights, when the force-applied (the pressure) is 2*lbs.* and the resistance 1*lb.*, the moving force would be 1*lb.*; but if the pressure were doubled, that is 4*lbs.*, the moving force would be 3*lbs.*; here, therefore, the ratio of the pressures would be as 1 to 2: but the ratio of the moving forces as 1 to 3, contrary to Mr. Whewell's theory: and the author submits, that the experiments brought forward in support of that theory, tend rather to maintain his own, for the reasons stated in chap. ix. § 4 to 10, part i.

In chap. x. part i. we have treated of the Composition of Forces; in which, besides the usual elementary propositions, we have introduced and demonstrated the proposition, that if two forces act at right angles to each other, their action, whether successive or simultaneous, will neither diminish nor increase each other; a proposition continually advanced and acceded to, but which the author is not aware has as yet been demonstrated. He has also added three other propositions, demonstrating the effect of forces acting together obliquely; that is, at an angle less or greater than a right angle: by one of which last three propositions (the last but one, p. 103.) the author has been enabled to demonstrate directly that the power of the screw, when turned by a force acting at its edge horizontally, that

is, at an angle with the direction of the spiral, is as the distance between the threads to the circumference of the cylinder in which the spiral moves. (See p. 164.) These demonstrations being geometrical, or synthetic, are more satisfactory, because more striking, than algebraic demonstrations of the same propositions, which are usually given by authors.

In Part II. of this work the author has treated of the six mechanical powers,—the Lever, Wheel and Axle, Pulley, Inclined Plane, Wedge, and Screw,—and has elucidated them by the demonstrations contained in the first part of the work. He has also added a chapter on the Ascent and Descent of Bodies over a Fixed Pulley.

In treating of the mechanical powers, the author has made it the basis of his reasonings, that forces (whether pressure or moving forces) acting simultaneously, produce the same effects as if they acted separately, and in succession; in which he has followed the example of Mr. Bridge: and this basis is warranted and established by experience. With respect to the lever, the author has considered the arcs which are described by the ends of the arms of the lever, as representing the velocities of bodies situated at the ends of the arms; for when the forces are equal, that is, in equilibrio, the weights are inversely as the velocities, and are also inversely as the arcs, which, therefore, may represent the velocities. And if this method does not demonstrate the power of the lever *à priori*, it is, at least, a familiar and intelligible illustration of it, and much less intricate than that of introducing the doctrine of the centre of gravity into the inquiry. And, in like manner, the author has treated the powers of the wheel and axle, and the pulley.

In treating of the inclined plane, the present author has demonstrated that the direction of its action is perpendicular to the plane, instead of *assuming* that property, as Mr.

Bridge has done (Mechanics, p. 273): by the aid of this elementary property, the other properties of the inclined plane, as well as those of the wedge and screw, (which are analogous to them,) are easily demonstrated by means of that beautiful proposition in the composition of forces,—that if a body be kept at rest by three forces, those forces will be represented by the three sides of a triangle drawn in the direction of the respective forces. (Art 17. ch. x. p. 90.)

In the seventh chap. part ii. the author has investigated anew the properties of the ascent and descent of bodies over a fixed pulley, to which he was led from finding the existing theory to be established on principles arbitrarily laid down, and not clearly defined; in fact, on new and insulated principles, apparently adopted for this particular subject, and applicable to no other. The author having, therefore, some doubt of the accuracy of the theory supported on so unsatisfactory a foundation, investigated it on the general principles which have been established by observation and uniform experience, and which are also applicable to other branches of the same subject. The result has produced a different theory, which the author now submits to the public, with this notice of the circumstances that gave rise to it, in order that it may not be supposed to be sanctioned by any previous authority. It was not, indeed, without much hesitation that the author ventured upon introducing this new theory, in opposition to the one supported by high authorities, and universally adopted for the last twenty-five or thirty years; but, as he further investigated the existing theory, he found it not satisfactorily supported by experiments, and, moreover, abounding in inconsistencies with itself, as is briefly noticed in chap. vii. part ii. of this work. The author apprehends that the existing theory cannot be traced back to the time of Archimedes, who was famous for his inves-

tigations of the properties of the pulley, and for the use he made of that powerful implement.

In order to render the work more intelligible to beginners, the author has demonstrated some propositions arithmetically, which might have been, perhaps, more directly demonstrated geometrically; for instance, the introductory lemma in chap. ii. might have been directly demonstrated from the first proposition of the sixth book of Euclid. But the author preferred the arithmetical demonstration which he has given of that lemma, because it is easy of comprehension to readers who are acquainted only with the common rules of arithmetic; and the geometric demonstration, to be understood, would require the reader to be conversant with the fifth book of Euclid, with which few beginners are very familiar. In general, it has been the author's endeavour to give the most easy demonstration that the case would admit of; with which view he has seldom, if ever, resorted to the technical method of algebra (which is often the rule without the principle), when he has had a choice, in demonstrating prime or important principles: and when reduced to this necessity, he has indicated every step in the algebraic process, by which the student may see the drift of the reasoning, instead of using those comprehensive assertions, by which the conclusion is arrived at by overleaping the intermediate steps, according to the practice of some authors, who seem to think that it is good at times to leave the learner to shift for himself. Whereby the student is set to discover and trace the sources and windings of those eastern streams which were explored five and twenty or thirty centuries ago by former adventurers, whose charts and journals are withheld from the student, or given to him in an obliterated state, or turned into enigmas for his benefit.

Throughout the work, the author has avoided confounding together *proportionality* and *equality* as identical;

which want of discrimination is a common defect of most (if not all) his predecessors. Mr. Bridge gives a notable instance of this practice in p. 8 of his work; in which, treating of uniform motion, he assigns the *ratios only*, which any one of the three quantities, the time, velocity, and space, bears to the other two; but in his first two examples (p. 8.) he gives the *values* of the spaces and the times; whereas his premises only warrant his giving their *ratios*, not their values. But when (in p. 9.) he assigns the ratios between the momenta, weights, and velocities of bodies moving uniformly, he is careful in the examples which he gives of his rules, to confine himself within the terms of his theory, and only gives their respective ratios. This discrepancy was doubtless owing to an overweening desire to maintain an uniformity between his two theories of uniform motion and momentum. He could not advance momentum beyond the state of ratio or proportionality, because the present state of science (for want of an easy experiment or two) does not admit of giving the quanta or values of the momenta of bodies; and, therefore (for the sake of uniformity), he reduces his theory of uniform motion to the same imperfect standard of proportionality, although its well-known properties give the *values*, not merely the *ratios*.

Some authors, indeed, expressly define momentum to be the product of the velocity and quantity of matter. (*Whewell's Mechanics*, § 167.) If quantity of matter here means weight; then, if a body of 1*lb.* weight moves at the rate of 16 feet per second, its momentum will be $16 \times 1 = 16$*lbs.* For the reason before mentioned, and more fully assigned in chap. vii. (*Momentum*) its momentum would be $= \frac{16}{d}$; where d denotes the velocity requisite to produce a momentum equal to the weight of the body; and if d is $= 4$ feet per second, the momentum of the

body would be $= \frac{16}{4} = 4lbs.$; and not $16lbs.$, as these authors suppose.

In presenting the results of these investigations to the public, the author feels that what is new or corrected, may be accepted as an accession, slighted as insignificant, or rejected as an innovation, and perhaps ridiculed as chimerical, according as the spirit of the age is disposed to the toils of further research, or to the repose of contentment with a moderate competence of this kind of knowledge; or as it rates its present attainments below, or at, the point of perfection. The author's expectations, so far as they are founded on the merit of the work itself, are not great; nor is he much encouraged by the apparent taste of the times, which seems bent rather on diffusing, than increasing or purifying, the golden stream of mathematical science.

Futurity, more propitious to the advancement of science, than the present age of phrenology, political economy, and bigotry to new prejudices, the successful rivals of old ones, may accept, with posthumous approbation, the patient, humble, and laborious endeavours of the author to discriminate and demonstrate just principles; to substitute correct ones for such as were mistaken; to reduce hazarded conjectures to certainty; to define what he found vague; to methodize what was desultory; to assign adequate causes to known effects; to present to view disregarded alternatives; to restrict unwarrantable inferences from experiments within their due bounds; to suggest truths as accessible to future experimentalists, where former ones may have failed for want of exactness, ingenuity, or perseverance, or, perhaps, of encouragement, prostituted to frivolous pursuits: in a word, to complete what the author found imperfect in this branch of science.

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A

NEW TREATISE

ON

MECHANICS.

CHAPTER I.

DEFINITION AND LAWS OF MOTION.

As motion is the passion of matter influenced by impulse, so a state of rest (which is the mere negation of motion) is the passion of matter not influenced by impulse.

First Law of Motion.—A body continues in its state of rest, or of uniform rectilinear motion, until, by some external force or impulse, its state is changed.

Second Law of Motion.—Motion, or the change of motion, is produced uniformly in the line of direction in which the impulse or force acts, and is proportional to the excess of the force applied above the resistance.

Third Law of Motion.—When a force applied to a body is resisted, the resistance reacts upon the body in a direction opposite to that of the force applied, and destroys *pro tanto* the action of the force applied.

[This law is generally stated as follows :—" Action and Reaction are equal, and in opposite directions."]

Explanation of the First Law of Motion.

Matter, abstractedly considered, cannot change its state, whether of rest or motion; when at rest it would continue at rest for ever, if not disturbed by some external force; and, when put in motion, it cannot of itself either stop or alter its motion. This inability to change its state of rest or motion is a property of matter called its inertia, vis inertiæ, or inert force.

That a body in motion would continue its motion uniformly in the same direction, but for some external force, is obvious from experiment and observation: for, a stone thrown along a sheet of smooth ice, skims along at first with nearly an uniform velocity; the motion, however, becomes gradually slower, and at last the stone stops. In this case it is obvious to our senses that the stone possesses the property of continuing its motion for a while, although with a velocity continually retarded. But if the stone be thrown along a flat surface, not so smooth as the sheet of ice, as, for instance, a flagged pavement, it will neither move so uniformly, nor so far; and, if upon a rougher surface, its motion will be slower, and cease still sooner. Now, we know from experience, that friction is an obstruction to the motion of a body moving along any surface, and that the friction is less in proportion as the surface is more smooth, and, therefore, we conclude, that the stone is obstructed and stopped by friction in each of these cases; the sooner, in proportion as the friction is greater; and that the obstruction is less in proportion as the friction is less; and, consequently, that if there were no friction at all, there would be no obstruction, and the stone would continue to move uniformly with its original velocity for ever. This friction is an external force occasioned by the contact of the stone with the surface on which it moves.

So, we observe that every heavy body has a tendency to fall downwards, if not supported; and that this tendency operates on bodies both at rest and in motion. Thus, the apple hanging on the tree would fall to the ground but for the resistance of the stalk which attaches it to the bough, which resistance is greater than the force which would impel it downward if it were loosened from the stalk, and, therefore, sustains the apple. This tendency to fall is gravitation, which causes all bodies projected horizontally near the surface of the earth to deviate from the line of projection into a curve towards the surface, to which it is, by the continued action of gravity, at length brought, and the motion is there destroyed by collision. As the projectile force is greater, the distance or space described by the body becomes greater, because the action of gravity is longer resisted; were the projectile force infinite, the action of gravity would be infinitely resisted; and were there no action of gravity, the projectile would move uniformly for ever, supposing there to be no other cause of obstruction. Here the external disturbing force is gravity.

Explanation of the Second Law of Motion.

This law is abundantly exemplified as to the direction of motion being that of the force impressed, by the aim which is taken in discharging missiles, in pointing the impulse in the direction in which the missile is designed to move. Thus, an arrow flies in the direction of the impulse imparted to it, except so far as its direction is changed by the force of gravity. The proportion of the motion to the impulse is treated of at large in Chapter IX. on Moving Force.

Explanation of the Third Law of Motion.

The third law of motion has relation to two bodies resisting each other, one of which only may be a moving body (which is, therefore, the acting body), the other body

(the one acted upon) being in a state of rest; or, both the bodies may be moving bodies.

1. In the first case, that in which one body only is a moving body and the other is at rest, the resistance of the body at rest consists of its weight and friction; if the weight and friction together are equal to, or greater than the force of the moving body, then the quantum of reaction of the resisting body is equal to the quantum of action of the moving body, and destroys it altogether. But, if the resistance of the body at rest is less than the action of the moving body, then the reaction of the body at rest is equal only to its own resistance, and is less than the action of the moving body, and destroys only so much of its action as is equivalent to the reaction. And in this manner is to be understood this law, as generally laid down by authors, viz. that "action and reaction are equal;" not that they are equal in point of quantum, but equal *pro tanto* in effect; as will be more apparent when we consider the case of two moving bodies acting on each other in opposite directions.

So, when it is said by Sir Isaac Newton, that when a horse draws a stone, the stone draws the horse as much in one direction as the horse draws the stone in the opposite one, we are not to conclude that the *quanta* of draught of the stone, and of the horse, are equal to each other (for, in that case, the horse would not move the stone from its place); but only that the quantum of the stone's reaction destroys an equal quantum of the horse's action; or, in other words, that action and reaction have equal *specific* forces in opposite directions: in short, that when the reaction is equal to the action, they balance, and mutually destroy each other; and, when the action exceeds the reaction, the reaction destroys an equal quantum of the action, leaving the balance of action unextinguished. The alternative case is impossible; viz. that of the reaction

exceeding the action, even when the resistance of the body at rest is greater than the force of the moving body. Hence, reaction and resistance, though akin to each other, are not identical. For when the resistance of a body at rest is greater than the action of the opposing force, the reaction of the resisting body is equal only to the action of the opposing force, and the excess of the resistance above the opposing force (and, therefore, above the reaction of the resisting body,) remains unemployed, otherwise than in keeping the resisting body at rest. So that the reaction of a resisting body can never exceed, but may be less than, its resistance; for reaction cannot exceed the opposing action, as the resistance may; and, therefore, universally, reaction is equal to the less of the two, viz. the action or the resistance.

2. The other case, of which the third rule admits, is when two moving bodies *act* upon each other. If the forces are equal, the action of each is opposed by an equal quantum of reaction of the other; and in this case it may be said, without any qualification whatever, that action and reaction are equal, and in opposite directions. But if the forces of the moving bodies are unequal, the reaction of the body of less force will only be equal to its own action, and not to the action of the other body, for its reaction cannot exceed its own force; and, therefore, it will only destroy so much of the action of the body of greater force as is equivalent to the action of the body of less force; so the reaction of the body of greater force will only be equal to the action of the body of less force; for the reaction cannot exceed the opposing action. For though an excess of force will remain in the body of greater force after impact, this excess will not be an excess of reaction, but an excess of force after the effect of the reaction is spent. In this modified sense is the expression, "Action and reaction are equal, and in opposite directions," under-

stood by Mr. Bridge and Mr. Whewell. "The moving force," says Mr. Bridge, "by which A communicates momentum to B, is called the *action* of A; and the tendency of B to diminish the momentum of A is called the *reaction* of B. Since, therefore, *according to this meaning of the words action and reaction*, the effect produced by the action of A is equal to the effect produced by the action of B;" "action and reaction are said to be equal during the impact of A and B." (*Bridge's Mechanics*, pp. 19, 20.) Mr. Whewell (*Mechanics, third edition*, p. 248) says:—"Momentum gained and lost, are sometimes called action and reaction; and, *in that case*, this proposition is true—reaction is equal and opposite to action." It is, however, much to be feared that the expression, "action is equal and opposite to reaction," is understood (by beginners, at least), almost in its fullest latitude, so as to make them believe, that if moving bodies of unequal forces oppose each other, their forces are equalised; and that the action of a force, resisted even by a less force, creates a reaction in the less force equal to the action of the greater; which absurdity even seems implied in the proposition that action and reaction are equal and opposite. For which reason the author has enunciated the third law of motion, so as to prevent any erroneous inference of this kind.

In general, the terms reaction and resistance are applied to the lesser of two forces acting in opposition to each other; and are applied to the body at rest, in the other case.

It follows, from the first law of motion, that motion, in its nature and essence, is uniform and perpetual; and that, when begun, it would never cease, but for the opposite action of an equivalent or greater force; which property of motion, if not controlled by a superior force, would evidently be very detrimental to the condition of the inhabitants, both rational and brute, of this earth, or to such like

inhabitants of any other sphere. Nay, it would be in the power of the inhabitants to transport themselves, or one another, from their world; or even to diminish, and at last destroy its fabric, by detaching parts from it, and casting off the fragments, to range for ever, without return, in the void expanse of space. To avoid these evil consequences, and, perhaps, for other wise and benevolent purposes, the Creator has limited and controlled this essential and inherent property of motion by gravity—an universal principle pervading matter in all its forms, and in every situation, by which all substances whatever are mutually attracted one to the other, and would fall or come against one another, but for some opposing force, or intervening substance. Thus, an apple hanging from a bough would, by the action of gravity, fall to the ground but for the stalk which sustains it; and every thing standing on the surface of the earth would fall to the centre but for the intervention of the matter about the earth's radii at those places.

The Creator has given to most, if not to all, animals the power of generating motion; and to mankind, through the aid of their reasoning faculties, the further power of projecting bodies with very great degrees of velocity: but the universal principle of gravity incessantly countervails these motions, (except when in the same direction with gravity, or in a direction inclined towards it,) and stops generally within a confined range, the motion originally generated, and requires new efforts to continue the motion, in the case of animals moving their own bodies, and a new impulse to repeat, and a continued impulse to prolong, any other generated motion.

But any principle which may act in opposition to that of gravity might restore the inherent property of motion, as expounded in the first law, if the opposing principle balanced that of gravity. For instance, magnetism *may* act in opposition to gravity, as it does in the case when a

magnet sustains by its magnetic force a piece of steel in contact under it; which, but for the magnetic force would by the force of gravity fall down in the opposite perpendicular direction; and if this opposing principle of magnetism could be brought to balance that of gravity, so that a ball of steel should remain suspended in the air at some point below the magnet, and, above the surface of the earth, in a state of equilibrium between the two opposite and there equal forces, the ball, if twirled about in that position, would, probably, suffer no impediment either from gravity or magnetism, but only from the friction of the contiguous particles of the atmosphere; and, therefore, it would seem that if the atmosphere were exhausted by an air-pump, the rotatory motion would not be impeded at all. Or, if the fable of the tomb of Mahomet could be realized, viz. that his remains are inclosed in a steel coffin, suspended in the air, in a cavern of adamant,—his steel coffin, if whirled round in vacuo, on any imaginary axis passing through its centre of gravity, would also (for any thing that appears to the contrary) continue to move in an endless rotation. For magnetism acts in a vacuum.

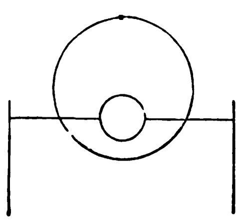
Magnetism is a confined and particular principle affecting only the magnet itself and some metals; but its action, like that of gravity, is attraction. The magnet and the metal attract each other mutually, like the sun and a planet, and when they can move more or less freely, they will meet at a point nearer the greater (or rather, the heavier,) of the two; as would, doubtless, be the case with the sun and any one of the planets, if the attraction of gravitation of the planet were not counteracted by the centrifugal force.

If gravity were abolished altogether out of nature, and all substances were magnetic, we may reasonably conclude that magnetism would effect then very nearly what gravity does in the existing state of things.

For let the sun be supposed to be a magnet, and the

earth a magnetic metal; also let the magnetic affection or attraction be equal to what the attraction of gravitation now is between the sun and the earth. If, under these circumstances, a force were communicated to the earth, equal to that which now keeps it in its orbit, and in the same direction, it would seem that the earth would describe its annual revolution under the influence of magnetism (which would then be the universal principle) in the same time and manner as it now does under the influence of gravity.

Ridiculous as the fable of Mahomet's tomb may appear, it is, however, no mean effort of the human imagination; and, indeed, it affords some important inferences. For if the exact place of suspension between the two opposing attractive forces of magnetism and gravity could be found and preserved in a vacuum; a rotatory motion generated in a ball suspended in equilibrio in that position must be continual, by the first law of motion, until it is stopped by some external force, which cannot be that of gravity or friction, for they are both excluded by the hypothesis: or, if a very light or hollow ball of magnetized steel could be made to revolve round a very powerful magnet in an orbit perpendicular to the horizon (in which case the magnet might be supported by two horizontal cords



at right angles with the plane of the ball's orbit); such an apparatus in a vacuum would exhibit the ball as an artificial planet revolving round the magnet as an artificial sun, and the revolutions of the ball round the magnet would be regular and perpetual (provided the atmosphere was completely exhausted), unless some external force (other than gravity or friction in the first case,) interfered with it. The author has not tried these experiments, but he apprehends that magnet-

ism is a power not tractable enough to be applied to such purposes; of which any person may satisfy himself who thinks that the experiments would succeed; only he is cautioned not to spend too much time over them. Assuredly, if a perpetual motion could not be maintained under such favourable circumstances (wherein gravity, friction, or collision, are not supposed to be obstructing forces,) it would be in vain to seek for it by other means in which one or more, or all, of these obstructing forces would offer a continual resistance.

CHAPTER II.

OF THE UNIFORM MOTION OF ONE BODY.

Definitions.

Uniform Motion is that motion by which a body moves over or describes equal spaces in equal times.

Velocity is the rate or degree of motion by which a body moving uniformly describes equal spaces in equal times.

The rate or degree of motion which denotes the velocity of a body moving uniformly, is usually expressed by the space which the body describes in 1″.

NOTATION.

The velocity of a body moving uniformly, the spaces which it describes, and the times in which those spaces are described, are denoted by the following letters or symbols: viz.

v, denotes the velocity.

s, the space described in any time.

t, *that* time.

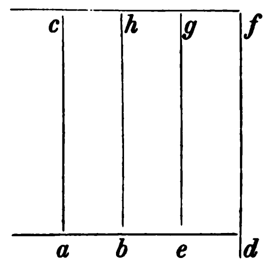
T, a greater or longer time than t.

S, the space described in the longer time, T.

Lemma.—The ratio of the spaces described by a body moving uniformly, is equal to, or the same as the ratio of

the times in which those spaces are described; that is, *the spaces are as the times*, or $s : S :: t : T$.

FIG. 1.



Let the right line *ab*, represent 1″, and the line *ad*, being made equal to three times *ab*, represent 3″: on *ab* describe any rectangle, *abhc*, representing the space described in 1″; and on *be* and *ed*, respectively, each being equal to *ab*, describe two other equal rectangles, *begh*, *edfg*, representing the spaces described in the second and third seconds respectively, and, consequently, each equal to the rectangle *abhc*; the whole rectangle *adfc*, equal to the three rectangles together, will represent the space described in the whole time, 3″; and the whole line *ad* will represent the whole time, 3″; and the right line *ac* will represent the uniform velocity. Hence, the whole rectangle *adfc* is three times as great as the rectangle *abhc*, and consequently the *spaces* represented by the rectangle *adfc*, and *abhc*, are as 3 to 1.

Also, the whole line *ad* is three times as great as the line *ab*, consequently the times represented by these lines are as 3 to 1, which we have shown to be the ratio of the spaces. Therefore (5 *Euclid* 11.) these ratios are equal to, or the same with each other, that is, the spaces are as the times:

and taking t = the line *ab*;
T = the line *ad*;
s = the rectangle *abhc*;
S = the rectangle *adfc*;

$$s : S :: t : T;$$

Corollary 1. $T = \frac{St}{s}$, by the Rule of Three;

Corollary 2. $$S = \frac{Ts}{t};$$

for $t : T :: s : S$; and by the Rule of Three $S = \frac{Ts}{t}$.

Theorem 1.

Any space is to the time in which it is described by a body moving uniformly, as any other space is to the time in which that other space is described by the body moving with the same uniform velocity; that is, $s : t :: S : T$.

For by the lemma $s : S :: t : T$;

Therefore, alternately, $s : t :: S : T$; (*See New Introduction to the Mathematics,* prop. 5. chap. xi.)

Coroll. 1. $t = \frac{sT}{S}$; for $S : T :: s : t$; $\therefore$ by the Rule of Three $t = \frac{sT}{S}$.

Coroll. 2. $s = \frac{tS}{T}$; for inversely $T : S :: t : s$; $\therefore$ by the Rule of Three $s = \frac{St}{T}$.

Theorem 2.

The space described by any body moving uniformly, is equal to the product of the velocity into the time: that is, $s = vt$.

For in figure 1, the line $c\,a$ is taken ad libitum; let it represent v, the velocity; and since it may be taken of any length, let it be such that the rectangle ca, ab, shall represent s, the space described in $1''$. Then, because that rectangle (*New Introduction to the Mathematics,* p. 28.) represents the product of the two factors denoted by the sides

ca, ab, that rectangle represents the product of the velocity into the time 1″.

So the equal rectangle, *hb. be,* represents the product of the velocity into the time, the second second; and the equal rectangle *ge. ed,* represents the product of the velocity into the time, the third second; and the whole rectangle *ca, ad,* (which is equal to the three rectangles *ca. ab; hb. bc;* and *ge. ed,* together,) will be equal to the sum of the products of the time into the velocity in each of the three seconds, that is, the sum of the products of all the parts; and because the sum of the products of all the parts is equal to the product of the whole (*New Introduction,* p. 15.) the rectangle *ca. ad,* is equal to the product of the whole time into the uniform velocity. Hence, universally, $s = vt$.

Theorem 3.

The time in which any body moving uniformly describes any space, is equal to the quotient of the space divided by the velocity; that is, $t = \frac{s}{v}$.

For we have seen in Theorem 2, that $s = vt$; hence t, alone, is not equal to s, but becomes equal to s when it is taken v times. Hence t, by itself, is the v^{th} part of s; that is, $t = \frac{s}{v}$.

Otherwise, algebraically;

since $s = vt$, by Theorem 2;

by dividing each of these equals by v, they become $\frac{s}{v} = t$.

Theorem 4.

The velocity with which any body moving uniformly describes any space, is equal to the quotient of the space divided by the time in which it is described; that is $v = \frac{s}{t}$.

For by Theorem 2, $s = vt$; wherefore, dividing both sides of the equation by t, $\frac{s}{t} = v$.

From these Theorems we collect the following formulæ: viz.

1 $S = \frac{Ts}{t}$; by the Lemma. Coroll. 2.

2 $T = \frac{St}{s}$; by ditto. Coroll. 1.

3 $t = \frac{sT}{S}$; by Theorem 1. Coroll. 1.

4 $s = \frac{tS}{T}$; by ditto. Coroll. 2.

5 $s = vt$; by Theorem 2.

6 $t = \frac{s}{v}$; by Theorem 3.

7 $v = \frac{s}{t}$; by Theorem 4.

By means of these formulæ, if any three of the four quantities s, S, t, T, be given, the fourth quantity may be found; and if any two of the three quantities s, t, v, be given, the third quantity may be found, as will appear in the following Table.

A Table of Formulæ for finding the Spaces described by Uniform Motions and their Velocities, and the Times or Duration of Motion.

	GIVEN.	SOUGHT.	VALUES.
1	T, s and t	S	$S = \frac{Ts}{t}$.
2	S, s and t	T	$T = \frac{St}{s}$.
3	S, s and T	t	$t = \frac{sT}{S}$.
4	S, T, and t	s	$s = \frac{tS}{T}$.
5	v and t	s	$s = vt$.
6	s and v	t	$t = \frac{s}{v}$.
7	s and t	v	$v = \frac{s}{t}$.

Problem 1. If a body moving uniformly has described 4 feet in 3″, in what time will the body describe 18 feet?

Here s is given $= 4$ feet; t, is given $= 3''$; and S, is given $= 18$ feet; and T, the time in which the greater space, 18 feet, is described, is sought.

Solution. By formula 2, $T = \frac{St}{s} = \frac{18 \times 3}{4} = 13\frac{1}{2}''$, the time required.

Problem 2. To find the space described; the velocity and time of motion being given.

Let a body move uniformly with a velocity of 16 feet per second for $11\frac{3}{4}''$; it is required to find the space which the body will have described in that time.

Here v is given $= 16$, and t is given $= 11\frac{2}{3}''$, and s is sought.

Solution.—By formula 5, $s = vt$; that is,

$$s = 16 \times 11\tfrac{2}{3} = 186\tfrac{2}{3} \text{ feet; the space required.}$$

Problem 3. To find the time in which any given space is described with a given velocity.

Let a railway train pass over a line of 100 miles, at the rate of 30 miles per hour when it is in motion; and let the train stop at five stations on the line for three minutes at each; at the end of this 100 miles let it stop a quarter of an hour, and then pass over another line of 120 miles at the rate of 28 miles per hour when going, and let it stop at six stations on *this* line for two minutes at each: It is required to find in what time the train will perform the whole journey?

Solution.

	Hours.	Min.
As to the first line (by formula 6) $t = \frac{s}{v} = \frac{100}{30} =$	3	20
As the second line, $t = \frac{120}{28}$	4	17 nearly
Stoppages on the first line, $5 \times 3 =$ ..	0	15
Stoppages on the second line, $6 \times 2 =$	0	12
Stoppage between the two lines $=$ ...	0	15
Total time of performing the journey..	8	19 nearly

Problem 4. To find the velocity, when the time and space are given.

Let a ship sail upon the equator in a course due west, from the twenty-fifth to the thirtieth degree of west longi-

tude, in 50 hours, with a steady breeze: It is required to find her average or mean rate of sailing?

Here t, the time, is given $= 50$ hours, and s, the space or distance performed, is given $= 30 - 25 = 5$ degrees $= 5 \times 60 = 300$ geographical miles, and v, the velocity, or mean rate of sailing, is sought.

Solution.—By formula 7, $v = \frac{s}{t} = \frac{300}{50} = 6$ geographical miles per hour, which is the mean rate of sailing that is sought.

Problem 5. With what velocities will a body moving uniformly describe 64 feet in 2″, 144 feet in 3″, 256 feet in 4″, 400 feet in 5″, and 576 feet in 6″?

In each of these cases s and t are given, and v, the velocity, is sought.

Solution.—By formula 7, $v = \frac{s}{t}$.

$$
\begin{aligned}
\text{Therefore, } v &= \frac{64}{2} = 32 \text{ feet per second, in } 2''. \\
&= \frac{144}{3} = 48 \quad \ldots \quad \ldots \quad \ldots \quad 3'' \\
&= \frac{256}{4} = 64 \quad \ldots \quad \ldots \quad \ldots \quad 4'' \\
&= \frac{400}{5} = 80 \quad \ldots \quad \ldots \quad \ldots \quad 5'' \\
&= \frac{576}{6} = 96 \quad \ldots \quad \ldots \quad \ldots \quad 6''
\end{aligned}
$$

which are the velocities sought.

Note.—The first two formulæ may (conveniently for proficients) be dispensed with out of the Table, if we consider the quantity sought as represented by its corresponding

italic letter; regulating the other letters accordingly. Thus, in the first problem we should use the third formula $t = \frac{sT}{S}$, representing the quantities $\frac{18 \times 3}{4}$; as before. But our object is to render the subject as intelligible as possible to beginners, by giving a distinct formula in the table for its appropriate case, instead of making one formula serve for both cases.

CHAPTER III.

OF THE MOTION OF TWO BODIES MOVING UNIFORMLY WITH DIFFERENT VELOCITIES.

Notation.

Let v represent the lesser velocity;
V the greater;
s the space described with the lesser velocity;
S the space described with the greater velocity;
t the time in which s is described;
T the time in which S is described.

General Theorem.

When bodies move uniformly with different velocities, they describe, from the necessity of this exigency, ***different*** spaces in ***equal*** times, and ***equal*** spaces in ***different*** times.

In the first case, viz. when different spaces are described in equal times, the comparison takes place ***between the spaces and the velocities.*** In this case ***the spaces are directly proportional to the velocities*** with which they are described; that is, $s : S :: v : V$; and *vice versâ.*

For let the lesser velocity be 3 feet per second, and the greater velocity be 6 feet per second, and let the time of motion be 2 seconds ($2''$), in that time the body moving with the lesser velocity would describe the space of $3 \times 2 = 6$ feet (formula 5, table, chap. ii.), and the body moving with the greater velocity would describe the space of $6 \times 2 = 12$ feet; which spaces 6 and 12 are directly as the velocities 3 and 6; as is obvious on inspection.

But in the second case, that is, when equal spaces are described in different times, the comparison lies *between the times and the velocities,* and *the times are in the inverse ratio of the velocities,* and *vice versâ*; that is, $t : T :: V : v$, (which is the inverse of $t : T :: v : V$,) and $t : T :: \frac{1}{v} : \frac{1}{V}$; also $v : V :: \frac{1}{t} : \frac{1}{T}$;

For taking the velocities as before, and the space described with both velocities to be 48 feet then (by formula 6, chap. ii.) $t = \frac{s}{v}$, and $T = \frac{S}{V}$;* that is, $t = \frac{48}{3} = 16$, and $T = \frac{S}{V} = \frac{48}{6} = 8$; hence $t : T :: V : v$ represents $16 : 8 :: 6 : 3$, which is true by inspection; so $t : T :: \frac{1}{v} : \frac{1}{V}$ represents $16 : 8 :: \frac{1}{3} : \frac{1}{6}$; and $v : V :: \frac{1}{t} : \frac{1}{T}$ represents $3 : 6 :: \frac{1}{16} : \frac{1}{8}$, both of which last proportions are true by inspection. For the reciprocals $\frac{1}{v}, \frac{1}{V}$, and $\frac{1}{t}, \frac{1}{T}$, are inversely as their denominators. (*New Introd.* p. 84.)

Hence we have (first expression) $t : T :: V : v$

(second expression) $t : T :: \frac{1}{v} : \frac{1}{V}$

(third expression) $v : V :: \frac{1}{t} : \frac{1}{T}$.

Otherwise,

By theorem 3, chap. ii. $t = \frac{s}{v}$, and $T = \frac{S}{V}$.

---

* Since $t = \frac{s}{v}$, whatever are the magnitudes of the quantities thereby denoted, $T = \frac{S}{V}$; for these capitals only denote that the magnitudes of the quantities which they represent are greater than the magnitudes of the quantities represented by $t = \frac{s}{v}$.

But by the hypothesis, $s = S$; and, therefore, S may be substituted for s in the equation, $t = \frac{s}{v}$.

Hence, $t = \frac{S}{v}$;

$$\text{and } t : \frac{S}{v} :: T : \frac{S}{V}; \text{ and, alternately,}$$

$$t : T :: \frac{S}{v} : \frac{S}{V};$$ and (dividing the last two terms by S) $t : T :: \frac{1}{v} : \frac{1}{V}$.

Hence, when the times are equal the spaces are *directly* as the velocities; that is $s : S :: v : V$; from which expression we can find any one of the four quantities, s, S, t, or T, the other three being given as follows:—

CASE I.

When the times are equal.

In this case (by the general theorem) $s : S :: v : V$.

Problem 1. Given S, v, and V, to find s.

Solution. Because the product of the extremes is equal to the product of the means of the proportionals, $s : S :: v : V$;

$sV = Sv$; and dividing both sides

by V, $s = \frac{Sv}{V}$; the value required.

Problem 2. Given s, v, and V, to find S.

Solution. $Sv = sV$;

and dividing by v $S = \frac{sV}{v}$; the value required.

Problem 3. Given s, S, and V, to find v.

Solution. $Sv = sV$;

and dividing by S $v = \frac{sV}{S}$; the value required.

Problem 4. Given s, S, and v, to find V.

Solution. $$sV = Sv;$$

and dividing by s $$V = \frac{Sv}{s};$$ the value required.

These four formulæ are collected in the following table for the sake of more ready reference.

TABLE I. CHAPTER III.

Table of Formulæ for finding the Spaces and Velocities of two bodies moving uniformly with different velocities for equal times.

	GIVEN.	SOUGHT.	VALUES.
1	S, v, and V	s	$s = \frac{Sv}{V}$
2	s, v, and V	S	$S = \frac{sV}{v}$
3	s, S, and V	v	$v = \frac{sV}{S}$
4	s, S, and v	V	$V = \frac{Sv}{s}$

CASE II.

When the spaces are equal.

In this case, by the general theorem,

$$t : T :: V : v; \text{ first expression.}$$

$$t : T :: \frac{1}{v} : \frac{1}{V}; \text{ second ditto.}$$

$$\text{and } v : V :: \frac{1}{t} : \frac{1}{T}; \text{ third ditto.}$$

From which expressions we can find any one of the four quantities t, T, v, or V, when the three others are given, in two distinct expressions for each value, by the following problems.

Problem 1. Let T, V, and v, be given: it is required to find t.

Solution 1. Because the product of the means is equal to the product of the extremes (see first expression),

$$tv = \mathrm{TV};$$

and dividing by v, $t = \frac{\mathrm{TV}}{v}$; one value of t.

Solution 2.—Because the product of the means is equal to the product of the extremes (see second expression),

$$t\frac{1}{\mathrm{V}} = \mathrm{T}\frac{1}{v};$$

and dividing by $\frac{1}{\mathrm{V}}$, $t = \frac{\left(\mathrm{T}\frac{1}{v}\right)}{\left(\frac{1}{\mathrm{V}}\right)}$; second value of t.

Note.—In this second solution, we use the second expression, not the third expression, because t, the sought quantity, is not involved in a fractional form in the second expression, which it is in the third.

Problem 2. Let t, V, and v, be given: it is required to find T.

Solution 1. Because, $\mathrm{TV} = tv$;

by dividing both sides by V, $\mathrm{T} = \frac{tv}{\mathrm{V}}$; one value of T.

Solution 2. Because, $\mathrm{T}\frac{1}{v} = t\frac{1}{\mathrm{V}}$;

by dividing by $\frac{1}{v}$; $\mathrm{T} = \frac{\left(t\frac{1}{\mathrm{V}}\right)}{\left(\frac{1}{v}\right)}$; 2nd value of T.

Problem 3. Let t, T, and v be given: it is required to find V.

Solution.—Because, $\quad TV = tv;$

dividing by T, $\quad V = \frac{tv}{T}$; one value of V.

Solution 2.—By the third expression, $v : V :: \frac{1}{t} : \frac{1}{T}$.

And because the product of the means is equal to the product of the extremes,

$$V\frac{1}{t} = v\frac{1}{T};$$

and dividing by $\frac{1}{t}$, $\quad V = \frac{\left(v\frac{1}{T}\right)}{\left(\frac{1}{t}\right)}$; second value of V,

by the third expression (which is here used for the reason stated at the foot of Prob. 1.)

Problem 4. Let t, T, and V, be given: it is required to find v.

Solution 1.—Because, $\quad tv = TV;$

(dividing by t) $\quad v = \frac{TV}{t}$; one value of v.

Solution 2.—Because, $\quad v\frac{1}{T} = V\frac{1}{t};$

dividing by $\frac{1}{T}$ $\quad v = \frac{\left(V\frac{1}{t}\right)}{\left(\frac{1}{T}\right)}$; the other value of v.

These last eight formulæ are collected in the following table:—

TABLE II. CHAPTER III.

Table of Formulæ for finding the Times and Velocities of two bodies moving uniformly with different velocities over equal spaces.

NO.	GIVEN.	SOUGHT.	VALUES.
1	T, V, and v	t	$t = \frac{TV}{v}$, first value; and $t = \frac{\left(T\frac{1}{v}\right)}{\left(\frac{1}{V}\right)}$, second value.
2	t, V, and v	T	$T = \frac{tv}{V}$, first value; $T = \frac{\left(t\frac{1}{V}\right)}{\left(\frac{1}{v}\right)}$, second value.
3	t, T, and v	V	$V = \frac{tv}{V}$, first value; $V = \frac{\left(v\frac{1}{T}\right)}{\left(\frac{1}{t}\right)}$, second value.
4	t, T, and V	v	$v = \left(\frac{TV}{t}\right)$, first value; $v = \frac{\left(V\frac{1}{t}\right)}{\left(\frac{1}{T}\right)}$, second value.

Note.—Bodies moving uniformly with equal velocities will always describe equal spaces in equal times; and, con-

sequently, in this case no question can arise requiring the aid of mathematical investigation to find the answer: for, if the space described by a body moving uniformly in a given time be given, the space that would be described by any other body moving uniformly with the same velocity in an equal time, is also given, because the spaces are equal: the times are also given in this case, for they must be equal, when the velocities are equal. But we are particularly to notice, that in the two cases that arise when the velocities are different (in the first of which cases the times are equal, and the spaces described, unequal; and in the second case, the equal spaces are described in different or unequal times), we can solve the different problems proposed in the first case; i. e. when the times are equal, without knowing *what* the times are; that is, without knowing more than that the times *are* equal. And in like manner we can solve the problems in the second case, when the spaces described are equal, without knowing *what* the spaces described are, but only that they *are* equal.

The reason why, in the first case, the knowledge of the values of the equal times is dispensed with, is because they being equal, the ratio of the spaces to the velocities is given; and that, in the other case, the knowledge of the values of the equal spaces is dispensed with, because they being equal, the ratio of the times to the velocities is given.

Examples in Case 1; *that is, when the Times are Equal.*

1. Let two bodies, A and B, move uniformly, A with the velocity of four feet per second, and B with the lesser velocity of three feet per second, and let A describe the space or distance of 144 feet in any time: it is required to find the space which B will describe in that time, or in an equal time?

Here S*v* and V are given, and *s* is sought.

Solution. By table i. formula 1,

$$s = \frac{Sv}{V} = \frac{144 \times 3}{4} = 108 \text{ feet; the answer.}$$

2. Let two bodies, X and Y, move uniformly, X with the velocity of four feet per second, and Y with the greater velocity of ten feet per second, and let X describe 108 feet in any time: what space will Y describe in that time, or in an equal time?

Here s, v, and V, given, and S is sought.

Solution. By table i. formula 2,

$$S = \frac{sV}{v} = \frac{108 \times 10}{4} = 270 \text{ feet; the answer.}$$

3. Let two bodies, M and N, move uniformly, and let M, with the velocity of $2\frac{1}{2}$ miles per minute, describe a space of 30 miles in any time; and let N describe a space of 6 miles in the same, or an equal, time. What is the velocity of N?

Here s, S, and V, are given, and v is sought.

Solution. By table i. formula 3,

$$v = \frac{sV}{S} = \frac{6 \times 2\frac{1}{2}}{30} = \frac{1}{2} \text{ mile per minute; the answer.}$$

4. Let two bodies, A and B, move uniformly, and let A, with the velocity of $\frac{1}{2}$ mile per minute, describe a space of $4\frac{1}{4}$ miles in any time, and let B describe the spaee of 34 miles in the same, or an equal time. What is the velocity of B?

Here s, S, and v are given, and V is sought.

Solution. By table i. formula 4,

$$V = \frac{Sv}{s} = \frac{34 \times \frac{1}{2}}{4\frac{1}{4}} = 4 \text{ miles per minute.}$$

Examples in Case 2, when the Spaces are Equal.

5. Let two bodies, C and D, move uniformly, and let C, with a velocity of 5 feet per second, describe any space in 20″, and let D describe the same space in 60″. What is the velocity of D?

Here the spaces are equal, and t, T, and V, are given, and v is sought.

Solution. By table ii. formula 4,

$$v = \frac{TV}{t} = \frac{20 \times 5}{60} = 1\tfrac{2}{3} \text{ feet per second;}$$

$$\text{and } v = \frac{\left(V\frac{1}{t}\right)}{\left(\frac{1}{T}\right)} = \frac{5 \times \frac{1}{60}}{\frac{1}{20}} = \frac{\frac{1}{12}}{\frac{1}{20}} = \frac{20}{12} = 1\tfrac{2}{3} \text{ ft. per second.}$$

6. Let two bodies, F and G, move uniformly, and let F, with a velocity of $\frac{1}{2}$ a foot per second, describe any space in $6\frac{1}{4}$ seconds, and let G describe the same, or an equal space in $3\frac{1}{8}$ seconds. What is the velocity of G?

Here t, T, and v, are given, and V is sought.

Solution. By table ii. formula 3,

$$V = \frac{tv}{T} = \frac{6\frac{1}{4} \times \frac{1}{2}}{3\frac{1}{8}} = 1 \text{ foot per second;}$$

$$\text{and } V = \frac{\left(v\frac{1}{T}\right)}{\left(\frac{1}{t}\right)} = \frac{\frac{1}{2} \times \frac{1}{3\frac{1}{8}}}{\frac{1}{6\frac{1}{4}}} = \frac{\frac{1}{6\frac{1}{4}}}{\frac{1}{6\frac{1}{4}}} = 1 \text{ foot per second.}$$

7. Let two bodies, H and I, move uniformly, and let H, with a velocity of $2\frac{1}{4}$ feet per second, describe any space in 5″, and let I, with a greater velocity, viz. $11\frac{1}{4}$ feet per second, describe the same, or an equal space. In what time will I describe that space?

Here t, V, and v, are given, and T is sought.

Solution. By table ii. formula 2,

$$T = \frac{tv}{V} = \frac{2\frac{1}{4} \times 5}{11\frac{1}{4}} = 1'', \text{ the time required;}$$

$$\text{and } T = \frac{\left(t\frac{1}{V}\right)}{\left(\frac{1}{v}\right)} = \frac{\left(5 \times \frac{1}{11\frac{1}{4}}\right)}{\left(\frac{1}{2\frac{1}{4}}\right)} = 1'', \text{ the time required.}$$

8. Let two bodies, K and L, move uniformly, K with a velocity of 25 feet per second, and L with a velocity of $2\frac{1}{2}$ feet per second, and let K describe any space in $5''$. In what time will L describe the same space?

Here T, V, and v, are given, and t is sought.

Solution. By table ii. formula 1,

$$t = \frac{TV}{v} = \frac{5 \times 25}{2\frac{1}{2}} = \frac{125}{2\frac{1}{2}} = 50'', \text{ the time required;}$$

$$\text{and } t = \frac{\left(T\frac{1}{v}\right)}{\left(\frac{1}{V}\right)} = \frac{\left(5 \times \frac{1}{2\frac{1}{2}}\right)}{\left(\frac{1}{25}\right)} = 50'', \text{ the time required.}$$

CHAPTER IV.

GRAVITATION OF BODIES NEAR THE SURFACE OF THE EARTH.

Art. (1.) It is ascertained by experiments that all heavy bodies, that is, such as are not materially affected by the pressure of the atmosphere, gravitate with equal degrees of velocity, without regard to the magnitudes of bodies of the same specific gravity, or to the differences between the specific gravity of other heavy bodies.

Gold, iron, coal, and ivory, are all heavy bodies, but of unequal specific gravities. One pound of gold gravitates with equal degrees of velocity, and in the same manner, as one ounce of gold; and so does one pound with one ounce of iron, and one pound with one ounce of coal, and one pound with one ounce of ivory. So one pound of ivory gravitates with an equal velocity with one pound of coal, with one pound of iron, and one pound of gold. So wood in a solid and compact state, being a heavy body, gravitates in the same manner as gold, iron, coal, and ivory, because it is not materially affected by the atmospheric pressure in that state; but wood, when not in a solid and compact state, as in shavings or saw-dust, being then a

light body, is materially affected by the atmospheric pressure, and is impeded in its gravitation in those states, in the same manner that gold in the leaf is impeded by it. But light bodies in a vacuum (from which the atmospheric air is excluded,) are found by experiment to gravitate with equal velocity with heavy bodies.

(2.) It has also been ascertained by experiments, that a heavy body will have descended, by the force of gravity, at or near the surface of the earth,

At the end of the		1st second	*about*		16 feet
..	..	2nd	..	..	64 do.
..	..	3rd	..	..	144 do.
..	..	4th	..	..	256 do.
..	..	5th	..	..	400 do.
..	..	6th	..	..	576 do.

and so on.

(3.) These distances, or spaces, being compared with the number of seconds, in which they are respectively described, are in the invariable ratio of the squares of the times.

(4.) For the space described in the first second is to the space described in the second second, as the square of one second is to the square of two seconds; that is,

$$16 : 64 :: 1^2 : 2^2 :: 1 : 4$$
$$\text{So } 64 : 144 :: 2^2 : 3^2 :: 4 : 9$$
$$\text{So } 144 : 256 :: 3^2 : 4^2 :: 9 : 16$$
$$\text{Also } 256 : 400 :: 4^2 : 5^2 :: 16 : 25$$
$$\text{And } 400 : 576 :: 5^2 : 6^2 :: 25 : 36$$

and so on.

(5.) Hence it appears that a heavy body gravitates

within the 1st second, *about*			16 feet.
.. .. 2d ..	..	$(64 - 16) =$	48 do.
.. .. 3d ..	..	$(144 - 64) =$	80 do.
.. .. 4th ..	..	$(256 - 144) =$	112 do.
.. .. 5th ..	..	$(400 - 256) =$	144 do.
.. .. 6th ..	..	$(576 - 400) =$	176 do.

and so on.

(6.) For since 64 feet is the space described *at the end* of two seconds, and 16 feet is the space described *at the end* of one second, the difference between these two spaces, that is $64 - 16 = 48$ must be the space described between the end of the first second, and the end of the second second, that is, *within* the second second; and so of the rest.

(7.) The spaces described by gravitation within any number of seconds, taking them successively from the beginning of the motion, increase in the ratio of the odd numbers, 1, 3, 5, 7, 9, 11, and so on:

Thus, $16 : 48 :: 1 : 3$

$48 : 80 :: 3 : 5$

$80 : 112 :: 5 : 7$

$112 : 144 :: 7 : 9$

and $144 : 176 :: 9 : 11$

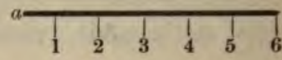
and so on.

[The word *about* is used in order to avoid noticing the fraction in the calculations; the actual space described in the first second being $16\frac{1}{10}$ feet nearly. See *Whewell's Mechanics,* 3d edit. article 224, p. 287.]

(8.) The *time* during which any gravitating body descends

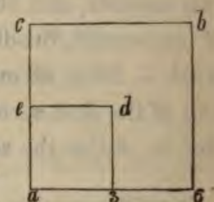
may be represented by a right line divided into as many equal parts as there are seconds contained in the time represented by the whole line.

Thus the right line $a\,6$, divided into six equal parts, may represent 6″, and each part, $a\,1$, $a\,2$, &c. may represent 1″.



(9.) The *spaces* described by a gravitating body in different times may be represented by squares described upon the right lines which represent the respective times.

(10.) Thus the square $a\,3\,d\,e$, described upon the right line $a\,3$, which represents 3″, and the square $a\,6\,b\,c$, described upon $a\,6$, would represent the spaces described by a gravitating body in 3″ and 6″, respectively; for those squares increase as the squares of the lines representing the respective times increase.



(11.) An *accelerated* motion is that by which the space described *within* each successive second is greater than the space described within the second next preceding it.

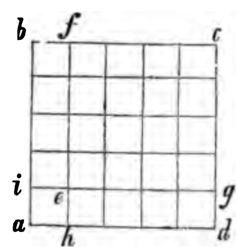
Thus, because by gravity 16 feet is the space described within the first second, 48 feet within the second second, 80 feet within the third second, and so on, the motion of gravity is an accelerated motion.

(12.) A motion is said to be *uniformly accelerated* when the excesses of the spaces described in each successive second are equal to each other; that is, when the difference between the spaces described *within* any two successive seconds is equal to the difference between the spaces described *within* any other two successive seconds.

(13.) The motion of gravitation is a motion uniformly accelerated. For the space described within the second second (48 feet) exceeds the space described within the first second (16 feet) by 32 feet, and the space described within the third second (80 feet) exceeds the space described within the second second (48 feet) also by 32 feet; and the excesses are equal also between any two other successive seconds. These excesses are each equal to double the space described in the first second, that is, $2 \times 16 = 32$.

(14.) The velocity acquired in the first second of descent by a body falling from rest, is 32 feet per second; for,

(15.) Let a body fall from rest for 1″, in which time it will descend through a space of 16 feet (represented by the small square, *a i e h*), and be then stopped for an instant, and then let fall again, and gravitate as from rest for 4″. Had it gravitated for the entire time, 5″, without interruption, it would have described the space represented by the square *a b c d*, in which the line *a b* represents 5″, that is, it would have described twenty-five of the squares, each equal to *a i e h*; but, in consequence of the stoppage at the end of the first second, it has described only the square, *a i e h*, in the first second, and the square *e f c g*, containing 16 squares, each equal to *a i e h*, in the last four seconds, equal together to seventeen times the square *a i e h*. Therefore, the loss of the velocity which it had acquired at the end of 1″ is represented by $25 - 17 = 8$ times the space represented by the small square *a i e h*, which is a loss of 8 small squares in 4″; and, consequently, of 2 small squares in each of the last 4 seconds; that is, a loss of 32 feet per second.



But this loss of 32 feet per second, during the last 4 seconds, is the loss of the velocity acquired by gravity in the first second; and, consequently, the velocity acquired by gravity in the first second of descent is 32 feet per second.

(16.) Hence the space (16 feet) described by gravity in the first second is half the space (32 feet), which would be described in 1″ by a body moving uniformly with the velocity acquired by gravity at the end of 1″.

(17.) In like manner it may be shown, that if at the end of 2″ the gravitating body were to lose its acquired velocity, it would describe within the next second 16 feet only, as from rest, in 1″; and, under these circumstances, it would describe in 3″, $64 + 16$ feet $= 80$ feet; but if it had retained its velocity acquired at the end of 2″, it would have described $(3 \times 3) \times 16 = 9 \times 16 = 144$ feet; consequently the velocity lost is $144 - 80 = 64$ feet, being the velocity acquired at the end of the second second.

(18.) Hence the space described by gravity in 2″ is half the space that would be described in 2″ by a body moving uniformly with the velocity acquired by gravity at the end of 2″; for, as we have seen, 64 feet is the space described by gravity in 2″; but a body moving uniformly with the velocity of 64 feet per second (the velocity acquired by gravity in 2″) would describe in 2″, $64 \times 2 = 128$ feet.

(19.) Hence the velocity acquired by gravity at the end of 2″ is equal to twice the velocity acquired by gravity at the end of 1″.

(20.) In like manner it could be shown, that

At the end of 3″, the acquired velocity is treble;

.. .. 4″ .. .. quadruple;

.. .. 5″ .. .. five times

the velocity acquired at the end of 1″, &c.; that is, *the*

velocity acquired at the end of any given time is equal to the product of the velocity acquired in 1″ into the given time.

(21.) Hence it follows that *the acquired velocities are directly as the times;* i. e. $t : T :: v : V$; for let the times be 2″ and 5″; the velocity acquired at the end of 2″ is (as we have seen) $32 \times 2 = 64$; and the velocity acquired at the end of 5″ is $32 \times 5 = 160$; and $2 : 5 :: 64 : 160$, as is obvious by inspection.

(22.) Hence also it follows that the space described by gravity in any given time, is half the space that would be described by a body moving uniformly for an equal time, with a velocity acquired by gravity at the end of the time.

(23.) Hence the velocities acquired by a gravitating body at the end of each second of its descent, are

At the end of the	1st	second	$16 \times 2 =$ 32	ft. per sec.
.. ..	2d	..	$32 \times 2 =$ 64	.. ..
.. ..	3d	..	$32 \times 3 =$ 96	.. ..
.. ..	4th	..	$32 \times 4 =$ 128	.. ..
.. ..	5th	..	$32 \times 5 =$ 160	.. ..
.. ..	6th	..	$32 \times 6 =$ 192	.. ..

and so on, increasing in an arithmetical progression, in which 32 is the common difference.

(24.) From these considerations we may define gravitation to be an accession of equal velocities in equal times; and we may also deduce the following theorems, formulæ, and rules, for ascertaining the spaces described by gravitating bodies, and their velocities, and times of descent.

Notation.

(25.) Let s denote the space described by a gravitating body in any number of seconds;

t the time or number of seconds in which the space s is described;

v denote the velocity acquired in describing the space s;

g denote 16 feet, being the space described by a gravitating body in the first second of its descent.

(26.) In consequence of the velocity acquired at the end of any given time being equal to the product of the velocity acquired in 1″ into the time (art. 20.), it follows, that if the time be a given quantity, the velocity may be found; for the velocity acquired in 1″ is also a given quantity = 32 feet = $2g$, that is, equal to twice the space described in the first second (2×16 feet); wherefore $v = 2gt$. It also follows, that if the time be given, the space described may also be found; for by art. 3, the spaces are as the squares of the times, and may, therefore, be found by the Rule of Three; wherefore, $1^2 : t^2 :: g : \frac{t^2 g}{1^2} = t^2 g = s$.

(27.) Again, if the velocity is given, both the time and the space may be found; for by article 21 the acquired velocities are directly as the times; and, therefore, the times may be found by the Rule of Three; that is, as the velocity acquired at the end of the first second is to the given velocity, so is 1″ to the time that is sought;

$$\text{Or, } 2g : v :: 1 : \frac{v}{2g} = t.$$

And the time being found, the space may be found by art. 26.

(28.) Also, if the space be given, the time and the velocity may be found. For (art. 3.) because the spaces are as the squares of the times, $g : s :: 1^2 : \frac{s \times 1^2}{g} = \frac{s}{g} = t^2$; wherefore, extracting the square root, $t = \left(\frac{s}{g}\right)^{\frac{1}{2}}$. And the time being thus found, the velocity may be found by article 26.

(29.) In the six following theorems we have enunciated and demonstrated the propositions of which the several formulæ are collected in the table in this chapter, for finding the values of any two of the quantities, s, t, and v, which may be required from any one of them which is given; premising to the student, that although only *one* of these quantities is *expressed to be* given, the quantity g is also a given or known quantity; so that, in reality, in each theorem *two* quantities *are* given, and it is required to find a third; which is worthy of notice for a full understanding of the subject.

Theorem 1.

(30.) The velocity is equal to the product of the time into twice the space described by gravity in 1″.

That is, $v = 2gt$.

This theorem is demonstrated in art. 26.; but the following demonstration is given as more strict and methodical.

Demonstration. By art. 2, the space described by gravity in 1″ is 16 feet $= g$; and by art. 23, the velocity acquired at the end of 1″ is $16 \times 2 = 32$ feet $= 2g$; and by art. 20, the velocity acquired at the end of any given time is equal to the product of the velocity acquired in 1″ into the given time, $= 2gt$;

Wherefore, $v = 2gt$.

Lemma.

(31.) The space is equal to half the product of the time into the acquired velocity;

That is, $s = \frac{1}{2}vt$.

Demonstration. For by art. 22, the space described by gravity in any given time, is half the space that would be described by a body moving uniformly for the same time with the velocity acquired at the end of the time. And by the rule of uniform motion (chap. ii. theorem 2,) the space uniformly described is equal to the product of the velocity into the time;

Wherefore, $s = \frac{1}{2}vt$.

Theorem 2.

(32.) The space is equal to the product of the square of the time into the space described by gravity in 1″.

That is, $s = gt^2$. (art. 26.)

Further Demonstration. For by the lemma, $s = \frac{1}{2}vt$..(A)

and by theor. 1, $v = 2gt$......(B)

Therefore, substituting $2gt$ for v, in equation (A),

$$s = \tfrac{1}{2}t \times 2gt, \text{ that is,}$$

$$s = gt^2.$$

Theorem 3.

(33.) The time is equal to the quotient of the velocity divided by twice the space described by gravity in 1″.

That is, $t = \frac{v}{2g}$. (art. 27.)

Further Demonstration. By theorem 1, $v = 2gt$;

and dividing both sides by $2g$, $\frac{v}{2g} = t$.

Theorem 4.

(34.) The space is equal to the quotient of the square of the velocity, divided by four times the space described by gravity in 1″.

$$\text{That is, } s = \frac{v^2}{4g}.$$

Demonstration. By theorem 3, $t = \frac{v}{2g}$;

and by squaring these equals, $t^2 = \frac{v^2}{4g^2}$.

And since by theor. 2,

$$s = gt^2;$$

by substituting $\left(\frac{v^2}{4g^2}\right)$ for its equal (t^2); in this last equation,

$$s = \frac{gv^2}{4g^2}; \text{ and dividing by } g,$$

$$s = \frac{v^2}{4g}.$$

Theorem 5.

(35.) The time is equal to the square root of the quotient of the space divided by the space described by gravity in 1″.

$$\text{That is, } t = \left(\frac{s}{g}\right)^{\frac{1}{2}}.$$

Demonstration. By theor. 2, $s = gt^2$;

and dividing by g, $\frac{s}{g} = t^2$;

and extracting the square root on both sides, $\left(\frac{s}{g}\right)^{\frac{1}{2}} = t$.

Theorem 6.

(36.) The velocity is equal to twice the square root of the product of the space into the space described by gravity in 1″.

That is, $v = 2 \times (gs)^{\frac{1}{2}}$.

By the lemma $s = \frac{1}{2}vt$, therefore, dividing by t,

$$\frac{s}{t} = \frac{v}{2}.$$

And because by theorem 1, $v = 2gt$;

by dividing by 2, $\frac{v}{2} = gt$.

Wherefore, $\frac{s}{t} = \frac{v}{2} = gt$.

And the product of any two of these equals is equal to the product of any other two; that is,*

$$\frac{v}{2} \times \frac{v}{2} = \frac{s}{t} \times gt; \text{ that is,}$$

$$\frac{v^2}{4} = \frac{gst}{t} = gs.$$

And by extracting the square root of each side of the equation,

$$\frac{v}{2} = (gs)^{\frac{1}{2}}; \text{ and multiplying by 2,}$$

$$v = 2 \times (gs)^{\frac{1}{2}}.$$

* This proposition is not demonstrated in Euclid or elsewhere; but it is in effect no more than an enunciation of his second axiom, that things which are equal to the same thing are equal to each other. For, to multiply equals by equals, is only to add equals together an equal number of times.

TABLE. CHAPTER IV.

A Table of Formulæ for finding the Spaces described by Gravitating Bodies during their times of descent, and their acquired velocities, and the times or duration of descent.

THEOREM.	GIVEN.	SOUGHT.	VALUES.
1	t	v	$v = 2gt$
2	t	s	$s = gt^2$
3	v	t	$t = \frac{v}{2g}$
4	v	s	$s = \frac{v^2}{4g}$
5	s	t	$t = \left(\frac{s}{g}\right)^{\frac{1}{2}}$
6	s	v	$v = 2 \times (gs)^{\frac{1}{2}}$

EXAMPLES.

(38.) 1. To find how far a heavy body will gravitate from a state of rest in 3″.

Here t, the time $= 3''$, and $g = 16$ feet are given, and s, the space that will be described in 3″, is sought.

Solution. By theorem 2, $s = gt^2 = 16 \times 9 = 144$ feet, which is the space described in 3″, being the space which a heavy body is found by experiment to describe by the force of gravity at the end of 3″. See article 2.

(39.) 2. To find the space through which a heavy body must fall from rest, before it acquires the velocity of 20 feet per second.

Here v, the velocity, $= 20$ feet, and $g = 16$ feet, are both given, and s, the space described, is sought.

Solution. By theorem 4,

$$s = \frac{v^2}{4g} = \frac{20 \times 20}{4 \times 16} = \frac{400}{64} = 6\tfrac{1}{4} \text{ feet,}$$

the space required.

(40.) 3. To find the time in which a heavy body will fall from rest through 90 feet, and the velocity which it will have acquired at the end of that time.

Here s, the space $= 90$ feet, and $g = 16$ feet are both given, and t and v, the time and velocity, are sought.

Solution. By theorem 5, $t = \left(\frac{s}{g}\right)^{\frac{1}{2}}$;

and by theorem 6, $v = 2 \times \left((gs)^{\frac{1}{2}}\right)$.

Hence $t = \left(\frac{90}{16}\right)^{\frac{1}{2}} = (5{\cdot}6)^{\frac{1}{2}} = 2{\cdot}36''$, the time required;

and $v = 2 \times \sqrt{16 \times 90} = 76$ feet per second, nearly; the velocity required.

(41.) 4. To find how far a heavy body will fall from rest in $2\frac{1}{2}''$, and the velocity acquired at the end of that time.

Here t, the time is given $= 2\frac{1}{2}''$, and g is known $= 16$ feet, and s and v, the space described and the velocity acquired, are sought.

Solution. By theorem 2, $s = gt^2$;

and by theorem 1, $v = 2gt$.

Therefore, $s = 16 \times (2\frac{1}{2} \times 2\frac{1}{2}) = 16 \times 6\frac{1}{4} = 100$ feet; the space sought.

And $v = 2 \times 16 \times 2\frac{1}{2} = 80$ feet per second; the velocity sought.

(42.) 5. To find the space which will have been described by a gravitating body from rest, at that instant when it has acquired the velocity of 160 feet per second, and to find the time in which it has acquired that velocity.

Here v, the velocity, is given $= 160$, and g is known $= 16$, and s and t, the time and space, are sought.

Solution. By theorem 4, $s = \frac{v^2}{4g}$;

and by theorem 3, $t = \frac{v}{2g}$.

Therefore, $s = \frac{160 \times 160}{4 \times 16} = \frac{25600}{64} = 400$ feet, the space;

And $t = \frac{160}{32} = 5''$; the time sought.

(43.) 6. To find the time in which a heavy body gravitating from rest will acquire, by the force of gravity, an equal velocity to that of a ball discharged from a cannon, taking the velocity at the time of discharge to be 704 feet per second, and to find the space which the heavy body will then have described.

Here v, the velocity, is given $= 704$ feet, and g, is a known quantity $= 16$, and t and s are sought.

Solution. By theorem 4, $s = \frac{v^2}{4g}$;

and by theorem 3, $t = \frac{v}{2g}$;

First, $s = \frac{704 \times 704}{4 \times 16} = \frac{495616}{64} = 7744$ feet; the space described.

Secondly, $t = \frac{704}{32} = 22''$, the time of gravitation.

(44.) 7. To find the spaces described, and the velocity acquired by a body gravitating from rest in the following times of its descent; viz.

TIMES.	gt^2	SPACES.	$2gt^2$.	VELOCITIES.
$\frac{1}{2}''$	$16 \times \frac{1}{4}$	$= 4$ feet	$32 \times \frac{1}{2}$	$= 16$ ft. per sec.
$\frac{1}{4}''$	$16 \times \frac{1}{16}$	$= 1$ ..	$32 \times \frac{1}{4}$	$= 8$ .. ..
$\frac{1}{8}''$	$16 \times \frac{1}{64}$	$= \frac{1}{4}$ ..	$32 \times \frac{1}{8}$	$= 4$ .. ..
$\frac{1}{3}''$	$16 \times \frac{1}{9}$	$= 1\frac{7}{9}$ ..	$32 \times \frac{1}{3}$	$= 10\frac{2}{3}$ .. ..
$\frac{1}{5}''$	$16 \times \frac{1}{25}$	$= \frac{16}{25}$ ..	$32 \times \frac{1}{5}$	$= 6\frac{2}{5}$ .. ..

As will be found by formulæ 2 and 1; for $s = gt^2$, and $v = 2gt$.

(45.) 8. To find the space through which a heavy body must fall before it acquires the velocity of 32 feet per second.

Here $v = 32$ feet, and $g = 16$ feet, are both given, and s is sought.

Solution. By theorem 4,

$$s = \frac{v^2}{4g} = \frac{32 \times 32}{4 \times 16} = \frac{1024}{64} = 16 \text{ feet; the answer,}$$

which agrees with article 2.

(46.) 9. To find the velocity which a gravitating body will acquire at the instant when it shall have descended 64 feet from a state of rest.

Here s is given, $= 64$ feet, and g is known, $= 16$ feet, and v, the velocity, is sought.

Solution. By theorem 6,

$v = 2 \times (gs)^{\frac{1}{2}} = 2 \times (1024)^{\frac{1}{2}} = 2 \times 32 = 64$ feet; the answer, which agrees with articles 2 and 23.

(47.) 10. To find the time in which a gravitating body will acquire the velocity of 128 feet per second.

Here $v = 128$ feet is given, and $g = 16$ feet is known, and t is sought.

Solution. By theorem 3,

$t = \frac{v}{2g} = \frac{128}{32} = 4''$; the answer, which agrees with article 23.

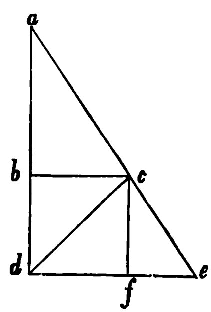
CHAPTER V.

GRAVITATION AUGMENTED BY VERTICAL IMPULSE.

Proposition 1.

THE ratios of the spaces described by a gravitating body in different times of its descent from a state of rest, may be represented by similar right-angled triangles, of which two homologous sides represent the times.

Let *a b c*, and *a d e*, be two similar right-angled triangles, and let *ab*, one of the sides of the triangle *abc*, represent one time, and let *ad*, the homologous side of the other triangle, represent the other time; the triangle *abc* shall be to the triangle, *ade* as the space described in the time *ab* is to the space described in the time *ad*.



Solution. By art. 3, chap. iv. the spaces are as the squares of the times; wherefore, the spaces are as the squares of the homologous sides, *ab*, *ad*, which represent the times; that is, $s : S :: ab^2 : ad^2$.

And (by proposition 36, *New Supplement to Euclid,*) the similar triangles *abc*, and *ade*, are also as the squares of the homologous sides, *ab*, *ad**; that is, triang. *abc* : triang. *ade* : : ab^2 : ad^2.

Wherefore (5 *Euclid*, 11.), *s* : S : : triang. *abc* : triang. *ade*.

The truth of the foregoing proposition may be put to the test of experiment as follows:—Let *ab* be equal to *bd*; *ab* will be half of *ad*; and because the triangles are equiangular (6 *Euclid*, 4.), *ac* will be half of *ae*, and *bc* will be half of *de*.

From the point *c*, draw the right line *cf* perpendicular to *de* (1 *Euclid*, 12.), and draw the diagonal *cd*.

Let *abc* and *ade* be the right angles in the right-angled triangles *abc* and *ade*; and because the exterior angle *abc* is equal to the interior and opposite angle *ade*, *bc* is parallel to *de* (1 *Euclid*, 28.); and, in like manner, it may be demonstrated, that *cf* is parallel to *ad*; wherefore, *bdfc* is a parallelogram, and the diagonal *cd* bisects it (1 *Euclid*, 34.); that is, the triangle *bcd* is equal to the triangle *cdf*. And because the triangles *cdf*, *cef*, are upon the equal bases *df*, *fe*, and between the same parallels *bc*, *de*, the triangles *cdf*, *cef*, are equal to each other (1 *Euclid*, 38.) And, in like manner, it may be demonstrated that the triangles *abc*, *cdb*, are equal to each other; wherefore, the whole triangle consists of the four equal triangles *abc*, *cdb*, *cdf*, and *cef*, and is quadruple of one of its parts, that is, of the triangle *abc*; hence, the triangle *abc* is to the triangle *ade* as 1 to 4.

* If the reader should not have the New Supplement to Euclid at hand, it may suffice to inform him, that the proposition here referred to, is there demonstrated from proposition 35 of that work; viz. that the squares of any two right lines are to each other in the duplicate ratio of the lines themselves, and because (by 6 *Euclid*, 19.) similar triangles are to one another in the duplicate ratio of their homologous sides, therefore, (5 *Euclid*, 11.) the similar triangles are to one another as the squares of their homologous sides.

But the spaces described in the times *ab*, *ad*, are as the squares of *ab*, *ad*; that is, as the squares of 1 and 2; i. e. as 1 to 4, which is the same as the ratio of the triangle *abc* to the triangle *ade*.

Corollary 1. The space described *within* any given time commencing from the beginning of the descent is represented by the right-angled triangle, of which the sides which include the right angle represent the time of descent, and the velocity acquired at the end of the time; and the space described *within* any subsequent time is represented by the trapezoid on the line which represents the subsequent time.

Thus, the triangle *abc*, described on the time *ab*, represents the space described *within* the time *ab*, commencing from the beginning of the descent; and the trapezoid *bdec* on the line *bd*, which represents the subsequent time of descent, represents the space described *within* that subsequent time.

Corollary 2. The velocity acquired by the gravitating body *within* such subsequent time of descent, is represented by the excess of the right line, which represents the velocity acquired in the whole time of descent, over and above the right line, which represents the previous time of descent.

Thus, the velocity acquired by the gravitating body *within* the subsequent time *bd*, is represented by *fe*, the excess of *de* over and above *bc*.

Proposition 2.

The ratios of the velocities acquired by a gravitating body in different times of its descent may be represented by any homologous sides *bc*, *de*, of the similar right-angled triangles, *abc*, *ade*, that is, $bc : de :: v : V$.

For the homologous sides *ab*, *ad* may represent the times of descent; and, because the triangles are equiangular, *ab* : *bc* : : *ad* : *de*, (6 *Euclid*, 4.) and alternately,

ab : *ad* : : *bc* : *de*;

And (by art. 21, chapter iv.)

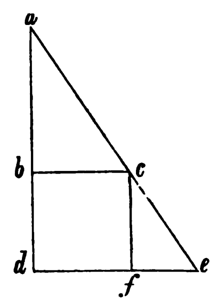
t : T; that is, *ab* : *ad* : : *v* : V;

Wherefore, (5 *Euclid*, 11.) *bc* : *de* : : *v* : V.

Proposition 3.

The space described by a gravitating body from any given point of its descent, during the remaining time of descent, is represented by the rectangle of the velocity acquired at the given point into the remaining time of descent, plus the space that would be described by the gravitating body falling from a state of rest within that remaining time.

The same construction being made, let *b* be the given point in the descent, then will S, the space described by the gravitating body from *b*, during the remaining time of descent *bd*, be represented by the rectangle *bc*. *bd*, plus the triangle *cfe*, which represents the space that would be described by the gravitating body falling from a state of rest within the remaining time *bd*.



Because *bc* represents the velocity acquired at *b*, the rectangle *bc*. *bd*, represents the space that would be described by the velocity *bc*, continued uniformly during the time *bd*. (Coroll. to lemma, chap. ii.)

And because *cf* and *ad* (by the construction) are parallel, the external angle *fce* is = the interior and opposite angle *bac*. (1 *Euclid*, 29.) And, in like manner, it may be shown that the angle *acb* is equal to the angle *cef*; and the angle

abc is a right angle (by the hypothesis), and the angle *cfe* is a right angle (by the construction); wherefore, the triangles *abc* and *cfe* are similar. But by proposition 1, $s : S :: \text{triang. } abc : \text{triang. } cfe$; and, alternately, $s : \text{triang. } abc :: S : \text{triang. } cfe$; and, because triangle *abc* represents *s*, triang. *cfe* represents S; the space that would be described by the gravitating body falling from a state of rest within the remaining time *bd*.

But the rectangle *bc*. *bd* + the triangle *cfe* constitute the trapezoid *bdec*, which (by coroll. 1, prop. 2.) represents S, the space described within the time *bd*.

Wherefore S is represented by the rectangle *bc*. *bd* + the triangle *cfe*.

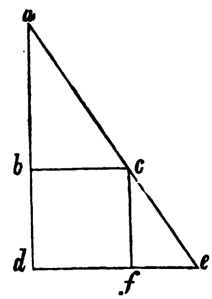
Coroll. 1. The velocity acquired at the point *b* is uniform.

Coroll. 2. The effect of the acquired velocity at any point *b*, during the time of descent *bd*, is the same as an equal velocity in the same direction communicated to the body when at rest at *b*. For the gravitating body describes from the point *b*, a space represented by the rectangle *bc*. *bd*, plus the triangle *cfe*, representing the space described by gravity in the time *bd*; and the velocity *bc*, communicated to the body at rest at the point *b*, would in the time *bd*, be also represented by the rectangle *bc*. *cd*, (coroll. to lemma, chap. ii.); and in the time *bd*, the body falling from rest at *b*, would describe a space represented by the triangle *cfe*.

From these general propositions, and their corollaries, we are enabled to deduce the following theorems and formulæ for solving the various problems which arise respecting the motion of gravitating bodies augmented by a vertical impulse.

Theorem 1.

The space described in a given time by a body impelled downward perpendicularly with any given uniform velocity is equal to the product of the velocity into the time, plus the space that would be described by gravity from rest within the given time.



Let *bd* be the given time, and *bc* the given velocity; then will S, the space described in the given time, *bd*, be equal to the rectangle *bc*. *bd*, + the triangle *cfe*.

The given velocity *bc*, is equal to the acquired velocity of the gravitating body at some point of time in its descent; let that point be *b*, in any part of the right-line *ad*, then will *bc* be equal to the acquired velocity at the point *b*, and the rectangle *bc*. *bd*, + the triangle *cfe*, will represent S, the space described *within* the time *bd*. And by coroll. to proposition 3, the effect of the acquired velocity at *b*, during the time of descent *bd* is the same as an equal velocity from rest perpendicularly downward; wherefore S = rect. *bc*. *bd*, + triang. *cfe*. Q. E. D.

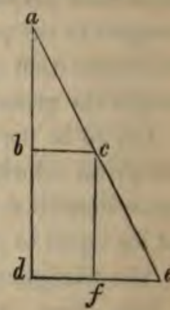
[This theorem gives the rule for solving problem 1, afterwards stated.]

Theorem 2.

The velocity which a body impelled downward perpendicularly for a given time, with a given initial velocity, would possess at the end of that time, is equal to the sum of the initial velocity, plus the velocity that would be acquired by gravity by a body falling from rest in the given time.

Let the line *bd* represent the given time, and let the line

bc represent the given initial velocity; then will the velocity possessed by the body at *d*, the end of the time, be equal to the initial velocity *bc*, plus the velocity that would be acquired by gravity by the body falling from rest at *b*, in the time *bd*.



Let the line *ab* represent the time in which the body falling from rest at the point *a* would acquire the velocity *bc*. From *a* draw *ae* through *c*; let *bc* be at right angles to *ad*, and from *d* draw *de* parallel to *bc*, meeting *ac* produced in *e*, and draw *cf* parallel to *ad*.

Because in the time *ab*, the body by gravity would acquire the velocity *bc*; triangle *abc* represents the space that would be described in the time *ab* (prop. 1.), and triangle *ade* represents the space that would be described by gravity in the time *ad*; and *de* represents the velocity acquired by gravity in the time *ad*.

But because *bc*. *bd*, is a rectangle, $df = bc, =$ the initial velocity; and by prop. 1, the triangle *cfe* represents the space that would be described from rest at the point *b* by gravity in the time *bd*, and *fe* (homologous to *bc*) represents the velocity acquired within the time *bd* (prop. 2.); wherefore, *de* ($= df + fe$) is equal to the initial velocity (*bc*) + the acquired velocity *fe*.

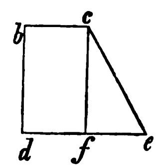
But (by prop. 1.) *de* is the velocity which the body would have acquired falling from rest at *a* in the time *ad*; and because the velocity acquired by gravity at the point *b*, is equal to the given initial velocity communicated at the point *b*, their effects are equal from the point *b* (coroll. 2. prop. 3.); wherefore, the velocity possessed by the body impelled from rest at *b* with the velocity *bc* is also equal to *de*, which has been demonstrated to be equal to the initial velocity *bc*, plus *fe*, the velocity that would be acquired

by gravity by the body falling from rest at *b*, in the time *bd*. Q. E. D.

[This theorem gives the rule for solving problem 2, afterwards stated.]

Theorem 3.

The space S, which is described by a body impelled perpendicularly downward with a given initial velocity, is = (the space that would be acquired by gravity from rest, in acquiring the excess of the total velocity at the end of the descent, over and above the initial velocity,) + (the space that would be described by the initial velocity from rest in the time that gravity acquires that excess).



Let *bc* (at right angles to *bd*) represent the initial velocity, and *bd* the time in which gravity would acquire from rest the velocity *fe* (= *de* − *df*), and let *de*, represent the total velocity at the end of the descent, that is, at the end of the time *bd*.

Join *ce*, *cf*, and because *cf* (= *bd*) is equal to the time in which gravity acquires the velocity *fe*, the triangle *cfe* represents the space described by gravity from rest in acquiring the velocity (*de* − *df*) = *fe* (coroll. 1, to prop. 1.) that is, in the time *cf*; and the rectangle *bf* is equal to the space described by the initial velocity *bc*, in the time *bd* or *cf*, in which gravity acquires the velocity *fe*; (coroll. to lemma, chap. 2.) i. e.

$$S = \text{rect. } bf + \text{triang. } cfe.$$

Wherefore, by theorem 1, S = rect. *bf* + triang. *cfe*.

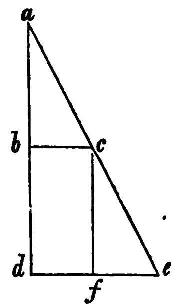
Wherefore, the space S, which is described by a body impelled, &c. Q. E. D.

[This theorem gives the rule for solving problem 3, afterwards stated.]

Theorem 4.

The initial velocity of any body impelled perpendicularly downward in any given time, is equal to the quotient of the difference between the whole space described in the descent, and the space that would be described by gravity in the given time of descent, divided by the given time of descent.

Let *bc* represent the initial velocity, and *bd* the given time of descent; and let *ab* represent the time in which gravity would acquire the velocity *bc*, and complete the figure in the margin.



By coroll. 1, to proposition 1, the space described within the time *bd* is equal to the trapezoid *bdec* = rect. *bdfc* + tri. *cef*, representing the space described by the uniform velocity, plus the space described by gravity.

Therefore, rect. *bdfc* = trapez. *bdec* — tri. *cef*; and rect. *bdfc* = *bc* × *bd* (*New Introduction*, chap. v. p. 28.); and (because (id. p. 33.) the quotient resulting from any product being divided by one of the factors is equal to the other factor);

$$bc = \frac{\text{rect. } bdec}{bd}. \quad \text{Q. E. D.}$$

[This theorem gives the rule for solving problem 4.]

Note.—In all the following problems in this chapter (v.) the small letters, *v* and *s*, apply to gravity; and the capital letters, V and S, apply to the initial velocity, and the small letter, *t*, denotes the time.

Problem 1.

To find the space that would be described by a body impelled perpendicularly downward with any given initial velocity, at the end of any given time of its descent.

Rule. By the table to chapter iv. formula 2, find the space that would be described by gravity from rest in the given time:

Multiply the initial velocity into the given time, the product will be the space described by the given velocity alone:

Add together these two spaces, their sum will be the whole space required.

[This rule is founded on theorem 1.]

EXAMPLE.

To find the space that would be described by a stone thrown perpendicularly down a high precipice with a velocity of 10 feet per second in 4″.

Here $V = 10$, and $t = 4$, are given, and $(S + s)$ are sought.

By the table, chap. 4, $s = gt^2 = 16 \times 16$, the space described by gravity in 4″, = ...	256
And by coroll. to lemma, chap. ii. $S = Vt = 10 \times 4$, the space described by the initial velocity in 4″ = ...	40
The total space described ..	296 feet;

the answer.

Problem 2.

To find the velocity which a body impelled downward perpendicularly for any given time of descent, with any given initial velocity, would possess at the end of the given time.

Rule.—Find by table, chap. iv. formula 1, the velocity that would be acquired by gravity from rest at the end of the given time:

To this velocity add the given initial velocity:

The sum will be the total velocity at the end of the given time.

[This rule is founded on theorem 2.]

EXAMPLE.

To find the velocity which a stone thrown perpendicularly down a high precipice, with an initial velocity of ten feet per second, would possess at the end of 4″ of its descent.

Here $V = 10$, and $t = 4''$, are given, and $V + v$ is sought.

By table, chapter iv. formula 1, v the velocity that would be acquired by gravity in the given time would be $= 2gt = 32 \times 4 =$ .. 128; the velocity acquired by gravity in 4″.

And $V =$ .. .. .. 10; the initial velocity

138 feet per second; the answer.

Problem 3.

To find the space which would be described by a perpendicular impulse downward, with a given initial velocity, the total velocity at the end of the descent being given.

Rule. Subtract the given initial velocity from the given total velocity, the remaining velocity will be the velocity acquired by gravity in the descent.

Find by the table in chap. 4, formula 4. the space that would be described by gravity in acquiring this velocity from rest:

And by formula 3, the time in which this velocity would be acquired by gravity from rest:

Multiply the given initial velocity into the time so found,

the product will be the space described by the initial velocity:

Add together the two spaces so found:

Their sum will be the space described in the descent.

[This rule is founded on theorem 3.]

EXAMPLE.

Let a stone be thrown down a well, with an initial perpendicular velocity of 10 feet per second, and at the bottom let its velocity be ascertained to be 170 feet per second: it is required to find the depth of the well from these data.

Here V is given = 10 feet, and $(V + v)$ is given = 170 feet, and the sum of $(S + s)$ is sought.

Since $V + v = 170$; $v = 170 - V = 170 - 10 = 160$ feet; the velocity acquired by gravity in the descent.

And by the table, chap. 4, $s = \frac{v^2}{4g} = \frac{25600}{64} = 400$ feet; the space described by gravity;

$$\text{and } t = \frac{v}{2g} = \frac{160}{32} = 5'';$$

And $10 \times 5 =$ .. .. 50 feet; the space described by the initial velocity.

Total space described .. 450 feet; the answer.

Problem 4.

To find the initial velocity downward; the space described in the descent, and the time of descent being given.

Rule. Find by the table, chapter iv. formula 2, the space that would be described by gravity in the given time of descent:

Subtract this space described by gravity from the given total space described in the descent; the remainder will be

the space described by the initial velocity in the time of descent:

Divide this remaining space by the time of descent:

The quotient will be the initial velocity required.

[This rule is founded on theorem 4.]

EXAMPLE.

Let a stone which is thrown with a perpendicular velocity down a well 450 feet deep, reach the bottom in 5″, what was the initial velocity with which the stone was thrown downward?

Here $S + s$ are given $= 450$, and t is given $= 5''$, and V is sought.

By the table, chap. iv. formula 2, $s = gt^2 = 16 \times 25 = 400$, the space that would be described by gravity in 5″.

And $450 - 400 = 50 = S$, is the space described in 5″ by the initial velocity, and $V = \frac{S}{t}$; (table, chap. 2.) $= \frac{50}{5} = 10$ feet per second; the initial velocity sought; the answer.

Note.—The two following problems cannot be solved by any of the properties of gravitation which we have hitherto established by geometrical or synthetic reasoning; but they are easily solved by the algebraic or analytic method, which we have used for that purpose.

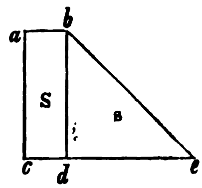
Problem 5.

To find the time of descent; the initial velocity, and the space described, being given.

Let the initial velocity be 32 feet per second; and let 384 feet be the whole space described: it is required to find the time of descent.

Here V is given $= 32$; and $(S + s)$ the whole space is given, $= 384$; and t, the time is sought.

Solution. Let the line ab represent V, the initial velocity, 32, and the line ac represent t, the time of descent;



Then will the rectangle $abdc$ represent the space S, described uniformly by the initial velocity ab, during the time of descent ac, which rectangle represents the space $32 \times t$;

Let the triangle bde represent s, the space described by gravity during the time of descent;

The trapezoid $abec$ will represent $(S + s)$, the sum of these spaces, that is, the whole space described $= 384$ feet;

And if the rectangle $(= 32 \times t)$ be abstracted from the trapezoid $abec$, the space described by gravity (represented by the triangle bde) will be equal to $384 - (32 \times t)$ which is, therefore, one expression of the value of s, that is, of the space described by gravity in the time t, in which expression t is the only unknown quantity.

By the table, chap. iv. formula 2, s, the space described by gravity in any time t, is equal to $gt^2 = 16 \times t^2$, which is, therefore, another expression of the value of s, in which expression t is again the only unknown quantity;

And because both these expressions are equal to s, the expressions are equal to each other; that is, $16 \times t^2 = 384 - (32 \times t)$; and transposing $(- (32 \times t))$; $16 \times t^2 + (32 \times t) = 384$; and (dividing both sides by 16),

$$t^2 + (2 \times t) = 24, \text{ that is, } t^2 + 2t = 24;$$

and *(by completing the square)*

$$t^2 + 2t + 1 = 24 + 1 = 25;$$

and, by extracting the square roots of these equal quantities,

$$t + 1 = \sqrt{25} = 5;$$

and transposing 1,

$$t = 5 - 1 = 4''\text{; which is the time sought.}$$

Problem 6.

To find the velocity acquired by gravity in the time of descent; the initial velocity, and the whole space described being given.

Rule. Find the time of descent as in the solution of problem 5.

Then by the table, chap. iv. formula 1, find the velocity that would be acquired by gravity in the time of descent; which is the velocity required.

EXAMPLE.

Let the initial velocity be 32 feet per second, and let 384 feet be the whole space described in the descent: it is required to find the velocity that would be acquired by gravity in the descent.

Here V is given $= 32$ feet, and $(S + s)$ the whole space is given $= 384$, and v is sought.

By problem 5, the time of descent is $4''$; and by the table, chap. iv. formula 1, $v = 2gt = 2 \times 16 \times 4 = 128$ feet per second; the acquired velocity sought.

CHAPTER VI.

GRAVITATION COUNTERACTED BY VERTICAL IMPULSE.

(Art. 1.) Heavy bodies may be impelled perpendicularly upwards with different degrees of force, that is, with different initial velocities. In these cases the bodies move upwards with a continually retarded motion. For as gravity acts in augmentation of a velocity imparted perpendicularly downward, it acts contrariwise in an equal degree in diminution of an opposite motion, until that motion is destroyed; when the ascent ceases, and the body falls down again by the force of gravity.

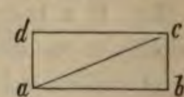
(2.) If a body were impelled with a given uniform velocity in one direction, and at the same moment by an equal uniform velocity in the opposite direction, the body would remain at rest; for these equal opposite forces mutually balance, and destroy each other.

(3.) If the contrary velocities being continued uniformly, would describe equal spaces in equal times; and if the times were taken *successively*, the space moved over in one direction, in any time, would be moved over again in the opposite direction in an equal time, and the effect upon the body moved at the end of the second time would be

the same as if the velocities were simultaneous; in which case, as we have seen in art. 2, the body would not move at all. Hence, if S and s represent any two spaces, when $V - v = 0$; $S - s = 0$.

(4.) If one of the contrary uniform velocities would in any given time move over half the space that the other velocity would move over in an equal time, then if the times were successive, half the space described in the first time with the greater velocity would be moved over in the opposite direction in the next successive equal time; and the effect would be, that at the end of that next successive equal time half the space only would remain that was described by the greater velocity in the first time. The effect would be the same as to the space, if these velocities were simultaneous, instead of being successive; but the velocity with which that remaining space would be described would be only half the greater velocity, for only half the space would be described when the velocities were simultaneous, that would be described by the greater velocity in an equal time.

(5.) If a body be impelled by a force generating an uniformly accelerated velocity for any time, and be then stopped, and impelled in the opposite direction with the acquired velocity continued uniformly, the space described with the uniform velocity in an equal time will be double the space described with the accelerated velocity.



For let the right line ab represent the time, bc the velocity acquired at b, and let the triangle abc, represent the space described with the accelerated velocity (coroll. 1. prop. 1. chap. v.); then will the rectangle $bcda$ represent the space described by the acquired velocity continued

uniformly in the equal time *cd* (lemma, chap. ii.), and (1 *Euclid* 34.) the rectangle *bcde* is double the triangle *abc*.

(6.) The result of these successive motions would be, that at the end of the second time the body would have moved from its original position in the direction of the uniform force, half the distance moved over with the uniform velocity.

For let the distance moved over by the accelerated velocity in the first time be represented by the right line *ab* in the direction from *a* to *b*; and let the distance moved over by the uniform velocity in the second time be represented by the right line $bc = 2ab$ in the opposite direction from *b* to *c*, the body will at the end of the second time come to the point *c*; that is, will have described the distance *ac* being half of *bc*, in the direction from *a* to *c*; and, therefore, $ac = ab$.

(7.) If these forces acted simultaneously, the effect with regard to the distance *ac* would be the same at the end of one time, as (if the forces acted successively) at the end of the second time; but the action in the direction from *a* towards *b*, would be destroyed by the greater action, in the direction from *a* towards *c*; and the greater action would be diminished by the quantum of the less action, and would only perform half the distance *ac* in the time in which, if not obstructed, it would perform the distance *bc*, consequently the body would not move at all from *a* towards *b* (for that tendency has been destroyed), but in that time would move from *a* to *c* with a velocity uniformly retarded by the uniformly accelerated opposite velocity.

(8.) Hence, if with the accelerated velocity the body would describe the space of 16 feet in one second, and with the uniform velocity would describe double that space, 32

feet in one second, by these forces acting simultaneously, the body would describe with a velocity uniformly retarded $32 - 16 = 16$ feet in one second.

(9.) Hence also, the velocity acquired by the uniformly accelerated motion in describing any distance is equal to a velocity which, if continued uniformly, would describe double that distance in an equal time.

(10.) Hence also, the body would cease to move in the direction of the initial uniform velocity at the end of an equal time to that in which the acquired velocity would become equal to the initial velocity.

(11.) Hence further, the space described by the joint effect of these motions is equal to the separate effect of the accelerated velocity alone; for ac is equal to ab, that is $bc - ab = ab$.

Theorem 1.

The initial velocity of a body impelled perpendicularly upward is equal to the velocity which would be acquired by a heavy body gravitating from rest in the time of ascent.

For the body is simultaneously impelled upward by the uniform initial velocity, and in the opposite direction by the action of gravity: which action generates an uniformly increasing velocity that would ultimately become equal to the initial velocity (chap. iv. art. 13.); and by art. 10, the body would cease to ascend at the instant when these velocities would become equal; and, consequently, the time of ascent is equal to the time in which gravity would acquire a velocity equal to the initial velocity; that is, the time of ascent is commensurate to the time of acquisition, because identical with it: wherefore, the initial velocity, &c. Q.E.D.

Theorem 2.

The space that would be described by a body impelled

perpendicularly upward is equal to the space that would be described by a body falling from rest during the time of the ascent.

Let the opposite forces be supposed to act in two equal successive times, each time being equal to the time of ascent; and, in the first time let gravity describe the space *ab*; in the next time the initial velocity acting uniformly would move double the distance of *ab* in the opposite direction

through the point *a* to *c* (art. 5.), *bc* being made the double of *ab*, and, consequently, *ac* being equal to *ab*: and the result of these opposite forces acting successively would be, that at the end of the second time the body impelled would have moved from *a* its original position, and would have described the space *ac*, equal to the space *ab* described by gravity in the first time.

But these opposite forces act simultaneously; in which case the effect is the same (art. 7.) as to the space described, as if the forces acted successively; wherefore, the height *ac* to which the body would ascend is equal to *ab*, the space which it would describe from rest during the time of ascent. Q. E. D.

Theorem 3.

The height to which a heavy body will ascend that is impelled upwards with a given velocity is equal to the space which the body would describe from rest at the instant of time when it acquired an equal velocity.

By theorem 2, the space that would be described by gravity during the time of its ascent is equal to the space described by the body in its ascent; and by theorem 1, the acquired velocity would then be equal to the initial velocity.

Wherefore, the height to which a heavy body will ascend that is impelled upwards, &c. Q. E. D.

Theorem 4.

The initial velocity of a heavy body impelled perpendicularly upward is equal to the velocity which a body falling from rest would acquire by gravity in describing a space equal to the height of ascent.

For by theorem 1, the initial velocity is equal to the velocity that would be acquired in the *time* of ascent; and by theorem 2, gravity in the time of ascent *ab* would describe a space equal to the height of ascent.

Wherefore, the initial velocity of a body impelled perpendicularly upward is equal to the velocity acquired by gravity in describing a space equal to the height of ascent.

Theorem 5.

The time during which any body impelled perpendicularly upward with any given velocity would continue to ascend is equal to the time in which the body would acquire an equal velocity by gravity from rest.

For by theorem 1, at the end of the time of ascent the velocity that would have been acquired by gravity in that time is equal to the initial velocity.

Wherefore the time of ascent is equal to the time in which the body would acquire a velocity by gravity from rest equal to the initial velocity.

Theorem 6.

The time in which any body impelled perpendicularly upward would continue to ascend is equal to the time in which a body falling from rest would describe a space equal to the height of ascent.

For by theorem 5, the time of ascent is equal to the time of acquiring by gravity a velocity equal to the initial velocity.

But by theorem 3, in the time of acquiring a velocity equal to the initial velocity, a body falling from rest would describe by gravity a space equal to the height of ascent.

Wherefore the time of ascent is equal to the time in which a body falling, &c. Q. E. D.

Note.—In the following problems the notation is the same as in the problems relating to perpendicular descent.

Problem 1.

To find the initial velocity of ascent, the time of ascent being given.

Rule.—By the table, chap. iv. find the velocity that would be acquired by gravity in the given time; this velocity (by theorem 1.) is equal to the initial velocity of ascent.

EXAMPLE.

If a body which is impelled perpendicularly upward continues to ascend for 4″, what is its initial velocity?

Here t is given $= 4''$, and V is sought, which by theorem 1 is $= v$.

By the table, chap. iv., $v = 2gt = 32 \times 4 = 128$ feet per second, the initial velocity.

Problem 2.

To find the height of ascent, the initial velocity being given.

Rule. By the table, chap. iv., find the space that would be described by gravity in acquiring the given initial velocity; this space (by theorem 3) is equal to the height of ascent.

EXAMPLE.

If a body is impelled perpendicularly upward with an

initial velocity of 192 feet per second, to what height will it ascend?

Here V (which, by theorem 4, is equal to the acquired velocity v) is given $= 192$; and $(S - s)$* is sought, which by art. 11, is equal to s.

By the table, $s = \frac{v^2}{4g} = \frac{36864}{64} = 576$ feet; which is the height of ascent.

Problem 3.

To find the height of ascent, the time of ascent being given.

Rule. By the table, find the space which gravity would describe in the given time; this space (by theorem 2) is equal to the height of ascent.

EXAMPLE.

If a body which is impelled perpendicularly upward continues to ascend for 4″, for what height will it ascend?

Here t is given $= 4''$, and $(S - s)$ is sought, which by theorem 2 is equal to s.

By the table, $s = gt^2 = 16 \times 16 = 256$ feet; which is the height of ascent.

Problem 4.

To find the initial velocity, the height of ascent being given.

Rule. By the table, find the velocity which gravity would acquire in describing a space equal to the height of ascent; the velocity so found is, by theorem 4, equal to the initial velocity.

EXAMPLE.

If a body which is impelled perpendicularly upward

* The expression $(S - s)$ means the space actually described by the initial velocity (S) when counteracted by gravity; that is, when diminished by (s).

ascends to the height of 256 feet, what was its initial velocity?

Here $(S - s) = s$ is given $= 256$ feet, and V is sought, which by theorem 4 is equal to v.

By the table, $v = 2 \times (gs)^{\frac{1}{2}} = 2 \times (16 \times 256)^{\frac{1}{2}} = 2 \times 64 = 128$ feet; the initial velocity required.

Problem 5.

To find the time of ascent, the initial velocity being given.

Rule. Find by the table the time in which gravity would acquire a velocity equal to the initial velocity; the time so found will (by theorem 5) be equal to the time of ascent.

EXAMPLE.

Let a body be impelled upward perpendicularly with an initial velocity of 128 feet; how long a time will it continue to ascend?

Here $V = v$ is given $= 128$ feet per second, and t is sought.

By the table, $t = \dfrac{v}{2g} = \dfrac{128}{32} = 4''$; the time of ascent.

Problem 6.

To find the time of ascent, the height of ascent being given.

Rule. Find by the table the time in which gravity would describe a space equal to the height of ascent, the time so found will (by theorem 6) be equal to the time of ascent required.

EXAMPLE.

If a body impelled perpendicularly upward ascends to the height of 256 feet, what is the time of ascent?

Here $(S - s) = s$ is given $= 256$ feet, and t is sought.

By the table, $t = \left(\frac{s}{g}\right)^{\frac{1}{2}} = \left(\frac{256}{16}\right)^{\frac{1}{2}} = 4''$; the time of ascent required.

Scholium. It is obvious from the properties of gravity counteracted, treated of in this chapter, that a body impelled upward perpendicularly from the surface of the earth with any velocity, will ascend and descend in equal times, and will possess a velocity at the end of its descent, equal to the initial velocity with which it was impelled upward.

CHAPTER VII.

MOMENTUM.

(ART. 1.) Momentum is the power of matter when it has become the recipient of impulse.

(2.) Matter not in motion has no force or power. If a man were floating on the surface of the water, between a ship of heavy burden and the pier of a harbour, and the ship was in a state of rest, he would sustain no injury, although one side of his body touched the ship, and the other side of his body touched the pier, any more than if the ship had no more weight than a feather. Or, if the person were in the water between two ships of great burden, but both in a state of rest, if a side of each ship *touched* him without *pressing* against him, he would suffer no harm; nor would he if the vessels so nearly in contact were both in motion, sideways, provided the following vessel did not move faster than the other. In each of these cases, the effect of the mere contact of the individual placed between such heavy bodies would be 0, or zero, because he would sustain no pressure. But if the ship were driven towards the pier, by the wind or a current, or the two vessels were driven towards each other, their pressure would probably be fatal to the individual in so perilous a

situation; for he would have to sustain the *pressure* of these bodies coming into collision. The violence of this pressure would be greater or less, in proportion as the motion or *velocity* of the bodies was more or less, and as the *weight* of the bodies coming in collision was more or less. This joint effect of the velocity and weight of bodies in motion constitutes their *momentum.*

(3.) Any one body is found to have different momenta according as it moves with different degrees of velocity. Thus, if a body is impelled with a velocity of 16 feet per second, it would have only half the momentum that it would have if the velocity were 32 feet per second, and only one-third of the momentum which it would have if the velocity were 48 feet per second; and so on. Hence *the momenta of one body* (that is, when the weight continues unvaried) *are directly as the velocities.*

(4.) So, if a body of one ton weight moves with a velocity of 16 feet per second, another body of double the weight, moving with an equal velocity, will have double the momentum of the lighter body; and so on. Hence, *when the velocities are equal the momenta are directly as the weights.*

(5.) But these experiments only establish the *ratios* of the momenta.

Theorem 1.

The momentum of a body, whatever be its weight, being $= 0$ in a state of rest, and being only generated by motion, it is to be considered, in the first place, what velocity or rate of motion will cause the body to possess a momentum equal to its weight. This velocity can only be discovered by experiment, which does not appear to have hitherto been tried. Suppose such an experiment to be made, it will probably be found, that with a very small

velocity communicated to the body its momentum will be less than its weight; let small increases of velocity be communicated to it, until that velocity is found which produces a momentum equal to the weight; and, for example, let this velocity be four feet in one second; then, if we consider this velocity as $= 1$, the product of the weight into that velocity will be equal to the momentum; thus, let the weight of the body be 6lbs., the weight into the velocity will be $6 \times 1 = 6$, the momentum. Upon this hypothesis, if the velocity were 8 feet per second, that is, the double of 4 feet per second, the momentum (by art. 3.) would be double of the momentum generated by the velocity of 4 feet per second; that is, it would be $= 6 \times 2 = 12$; and if the velocity were trebled (12) the momentum would be trebled $= 6 \times 3 = 18$; and so on. Hence, we may consider the actual velocity as consisting of two parts when it generates a momentum exceeding the weight; viz. first, the velocity required to make the momentum equal to the weight, (which, by way of distinction, is here called the acquiring velocity); and, secondly, the velocity by which the momentum is made to exceed the weight (which is here called the augmenting velocity). It follows, that the augmenting velocity is the quotient of the actual velocity divided by the acquiring velocity; for taking the acquiring velocity to be four feet, the momentum of the body would be doubled, tripled, quadrupled, &c., according as the number four was contained twice, thrice, four times, &c. in the whole actual velocity; so let the actual velocity be 7 feet per second, $\frac{7}{4} = 1\frac{3}{4}$, will be the augmenting velocity; and the momentum of a body of 6lbs weight would (according to the above hypothesis) be $6 \times 1\frac{3}{4} = 10\frac{1}{2}$. Whence, it follows universally, that *the momentum is equal to the product of the weight into the actual velocity, divided by the acquiring velocity.*

Therefore, let M represent the momentum, V the whole actual velocity, W the weight, and let d represent 4 feet (being the supposed acquiring velocity);

$$\text{Then } M = W \times \frac{V}{d} = \frac{WV}{d}.$$

From this equation we obtain the *expressions* of the values (though not the actual values until the value of d be ascertained) of the weight, actual velocity, and augmenting velocity.

Let Y represent the augmenting velocity, $Y = \frac{V}{d}$; $\therefore M = WY$.

Theorem 2.

The weight of any body moving uniformly is equal to the quotient of the momentum, divided by the augmenting velocity; that is, $W = \frac{M}{Y}$.

Demonstration. Since $M = WY$;

by dividing these equals by Y, $\frac{M}{Y} = W$.

Theorem 3.

The actual velocity of any body moving uniformly is equal to the product of the quotient of the momentum divided by the weight into the acquiring velocity; that is,

$$V = \frac{M}{W} \times d = \frac{Md}{W}.$$

Demonstration. Since $M = \frac{WV}{d}$, by theorem 1,

by dividing these equals by W, $\frac{M}{W} = \frac{V}{d}$;

and multiplying by d, $\frac{Md}{W} = V$.

Theorem 4.

The augmenting velocity of any body moving uniformly is equal to the quotient of the momentum divided by the weight; that is, $Y = \frac{M}{W}$.

Demonstration. By theorem 1, $M = WY$;
and dividing by W, $\frac{M}{W} = Y$.

Theorem 5.

The reciprocal of the acquiring velocity of any body moving uniformly is equal to the quotient of the momentum divided by the product of the weight into the actual velocity; that is, $\frac{1}{d} = \left(\frac{M}{WV}\right)$.

Demonstration. By theorem 1, $M = \frac{WV}{d}$;
and dividing by WV, $\frac{M}{WV} = \frac{1}{d}$.

From these formulæ we might determine the momentum of a body moving uniformly, if the value of *d*, the constant divisor, representing the acquiring velocity was known; to ascertain which appears to be a great desideratum in science, inasmuch as without it, there is no satisfactory measure of momentum extant. The present author found its *ratio* ascertained, and he has advanced no further than to find the *expression* of its value, but in quantities of which one, viz. *d*, is as yet not ascertained.

CHAPTER VIII.

OF THE MOMENTA OF TWO BODIES CONSIDERED COMPARATIVELY.

Two bodies of equal weight, and with equal velocities, (that is, if W and V are equal in each,) will have equal momenta; and if W and M are equal, or if V and M are equal in each; then their velocities in the first case, and their weights in the other will be equal. Hence, if they are equal in any two of these respects, they are equal in the third.

They may also be equal in one respect, and unequal in both the others. Their momenta may be equal; and if their weights are unequal, so will be their velocities, and *vice versâ.* So also their weights may be equal, and their velocities and momenta unequal; and again their velocities may be equal, and their weights and momenta unequal. Or, they may be unequal in all respects.

Let M, W, and V, represent the momentum, weight, and velocity, of any body moving uniformly; and m, w, and v, the momentum, weight, and velocity, of any other body moving uniformly.

Theorem 1.

If two bodies of equal weight move uniformly with different velocities, their momenta will be as their velocities; that is, $M : m :: V : v$.

By theorem 1, chap. vii. $M = \frac{WV}{d}$, and $m = \frac{wv}{d}$; wherefore, $M : m :: \frac{WV}{d} : \frac{wv}{d}$; and multiplying the last two terms by d, $M : m :: WV : wv$; and by the hypothesis $W = w$, and substituting W for w, in the fourth proportional, $M : m :: WV : Wv$; and dividing the third and fourth proportionals by W; $M : m :: V : v$. Q. E. D.

Theorem 2.

If two bodies of different weights move uniformly with equal velocities, their momenta will be as their weights; that is, $M : m :: W : w$.

By the demonstration of theorem 1, $M : m :: WV : wv$; and by the hypothesis $V = v$, and substituting V for v in the fourth proportional, $M : m :: WV : wV$; and dividing the third and fourth proportionals by V, $M : m :: W : w$. Q. E. D.

Theorem 3.

If bodies of unequal weights, moving uniformly, have equal momenta, their velocities will be inversely as their weights; that is, $W : w :: v : V$.

By theorem 1, $M : m :: WV : wv$; $\therefore Mwv = mWV$; but $M = m$ by the hypothesis; and by substitution, $Mwv = MWV$; and dividing by M, $wv = WV$; $\therefore w : W :: V : v$; and inversely $W : w :: v : V$. Q. E. D.

Theorem 4.

If bodies, moving uniformly with unequal velocities, have equal momenta, their weights will be inversely as their velocities, i. e. $V : v :: w : W$.

This proposition is demonstrated in the last proposition.

From these four theorems we obtain formulæ for finding the momenta, weights, and velocities of bodies moving uniformly in the three following cases; viz. first when the weights only are equal; and secondly, when the velocities are equal; and thirdly, when the momenta only are equal.

CASE I.

When the weights are equal.

Problem 1.

Given M, V and v: it is required to find m.

By theorem 1, $M : m :: V : v$; and because the products of the extremes and means are equal, $Mv = mV$; and dividing each of these equals by V, $\frac{Mv}{V} = m$.

Coroll. $Mv = mV$.

Problem 2.

Given M, m and V, to find v.

By coroll. problem 1, $Mv = mV$; and dividing both sides by M, $v = \frac{mV}{M}$.

CASE II.

When the velocities are equal.

Problem 1.

Given M, W and w: it is required to find m.

By theorem 2, $M : m :: W : w$; and because the

products of the extremes and means are equal, $Mw = mW$; and dividing these equals by W, $\frac{Mw}{W} = m$.

Coroll. $Mw = mW$.

Problem 2.

Given M, m and W, to find w.

By coroll. problem 1, $Mw = mW$, and dividing by M,

$$w = \frac{mW}{M}.$$

CASE III.

When the momenta are equal.

Problem 1.

Given W, V and v, to find w.

First Solution. By theorem 3, $W : w :: \frac{1}{V} : \frac{1}{v}$; and because the products of the means and extremes are equal,

$$W\frac{1}{v} = w\frac{1}{V};$$

and dividing these equals by $\frac{1}{V}$; $\frac{W\frac{1}{v}}{\frac{1}{V}} = w$.

Second Solution. By theorem 3, $W : w :: v : V$; and because the products of the means and extremes are equal,

$$WV = wv;$$

and by dividing by v, $\frac{WV}{v} = w$.

Coroll. $WV = wv$.

Problem 2.

Given W, w and V, to find v.

First Solution. By theorem 4, $V : v :: \frac{1}{W} : \frac{1}{w}$; and because the product of the extremes is equal to the product of the means, $V\frac{1}{w} = v\frac{1}{W}$;

and dividing these equals by $\frac{1}{W}$, $v = \left(\frac{V\frac{1}{w}}{\frac{1}{W}}\right)$.

Second Solution. By coroll. problem 1, $WV = wv$; and dividing each side by w, $\frac{WV}{w} = v$.

Note. The second method of solution of these last two problems is more simple and better than the first method in each.

TABLE. CHAPTER VIII.

Formulæ for finding the momenta and velocities of bodies of equal weights moving uniformly.

	GIVEN.	SOUGHT.	VALUES.
1	M, V and v	m	$m = \frac{Mv}{V}$
2	M, m, and V	v	$v = \frac{mV}{M}$.

Formulæ for finding the momenta and weights of bodies moving uniformly with equal velocities.			
	GIVEN.	SOUGHT.	VALUES.
3	M, W, and w	m	$m = \frac{Mw}{W}$.
4	M, m, and W	w	$w = \frac{mW}{M}$.
Formulæ for finding the weights and velocities of bodies having equal momenta.			
	GIVEN.	SOUGHT.	VALUES.
5	W, V, and v	w	$w = \frac{WV}{v} = \frac{W\frac{1}{v}}{\frac{1}{V}}$.
6	W, w, and V	v	$v = \frac{WV}{w} = \frac{V\frac{1}{w}}{\frac{1}{W}}$.

Note. In these and the subsequent formulæ, the capital letters are used as applicable to those bodies of which two of the qualities are given; and the italics to those bodies of which only one quantity is given, and the other is sought.

It is also very material to observe, that in the first four cases comprised in the above table, either M or m, or both, are supposed to be found or given; and, consequently, until this is the case, that is, until the constant multiplier d, which represents the acquiring velocity is ascertained by experiments, the above table only gives the *expressions* of

the values in those four cases, not the values themselves; but in the last two cases in the table, where it is known that the momenta are *equal*, although the *values* of the momenta may be unknown, the weights and velocities may be found by the formulæ, as well as if the values of the momenta were ascertained by experiment or otherwise.

Many authors state in general, that the momentum is equal to the weight into the velocity; that is, $M = WV$, which will lead to a correct result when momenta are compared together, but not otherwise. This, therefore, is confounding *proportionality* with *equality*, or substituting $M = WV$ for $M \propto WV$, a practice too common with mathematicians. (*Gregory's Astronomy*, p. 177, *note*.) This confusion is owing to the common mistake of using the word *weight* as denoting *quantity of matter*, which when the body is moving is without weight.

Finally, because (theor. 1, chap. vii.) $M = W \times \frac{V}{d}$; which is $= \frac{W}{d} \times V$; it follows, that $\frac{W}{d}$ represents the quantity of matter without weight; that is, when the body is in equilibrio. Hence, $W - \frac{W}{d}$, expresses the difference between W, (the state or weight of the body when at rest,) and $\frac{W}{d}$, (its state when in equilibrio); and, consequently, $W - \frac{W}{d}$ denotes the weight abstracted from the quantity of matter; that is, the weight which the body loses in being brought into a state of equilibrium.

CHAPTER IX.

MOVING FORCE.

(Art. 1.) Matter at rest, being inert and inactive of itself, can only be put in motion by external means; and, whatever these means may be, they become force-applied.

(2.) Every body at rest is nevertheless acted upon by gravity as a force-applied which would impel it perpendicularly downward but for its support; and if the body is kept in equilibrio, it must be occasioned either by an equal force acting in an opposite direction (such as a force acting by a pulley) upwards, or by an obstruction or resistance, which exerts a reaction equal to the moving force of the body (such as the earth supporting the body).

(3.) If a force applied to a body at rest meets with a constant resistance from the friction of the body; and if the reaction of the resistance is less than the applied force, then (by the second law of motion) the body will be moved forward by the excess of the applied force, over and above the reaction of the resistance, in the direction of the original force

applied; and this excess will constitute its *moving force.* For, since by the third law of motion the reaction of the resistance destroys an equal quantum of the action of the applied force, it is the excess only of the action of the applied force above the reaction which moves the body. Hence, if the force-applied were double the resistance, the moving force would be half of the force-applied; and if the force-applied were triple the resistance, the moving force would be double the resistance, and if the force-applied were quadruple the resistance, the moving force would be triple the resistance; and so on.

Thus, if the force applied be $= 1$, the resistance $= \frac{1}{2}$; then $1 - \frac{1}{2} = \frac{1}{2}$, the moving force; and if the force applied $= 2$, the quadruple of the resistance $= \frac{1}{2}$; then $2 - \frac{1}{2} = 1\frac{1}{2} = \frac{1}{2} \times 3$, that is, when the force-applied is *quadruple* the resistance, the moving force is *triple* the resistance; and so on.

(4.) *The moving force is as the cause of which the momentum is the full and entire effect.* Thus momentum is moving force transformed into a different state; a change inferring neither loss nor increase. For momentum may be transformed back again into its original state of moving force, and that also without loss or increase. Thus, if a body, moving with any velocity, comes in contact with a non-elastic body at rest, of equal weight, the two bodies, after impact, will proceed with half the original velocity in the same direction; which result is obviously the same as if the moving force by which the first body was impelled operated upon both bodies at once at the time of impact. In this instance of re-transformation, an equal quantum of moving force is produced, but not in the same plight; for the retransformation is, in fact, of only half of the momentum, the other half of the momentum continuing in

the body first moved. But if the whole momentum of the first body were communicated to a *Spring*, this spring being let loose would reproduce the moving force in its original plight; that is, so as to produce an equal velocity on another body of the same weight.

(5.) It follows, therefore, that moving force is not only proportional to momentum, but equal to it; and, consequently, we may adopt the same formulæ for moving force which are contained in the table, chap. viii. with regard to momentum, substituting F, the symbol for moving force, instead of M in the formulæ.

(6.) But as it does not seem altogether settled among mathematicians (at least, it is not distinctly stated,) of what moving force actually consists, it may be worth while to consider the matter somewhat further.

(7.) It appears from experiment, that if a weight of 3*lbs.* were placed on a perfectly smooth horizontal table, and another weight of 1*lb.*, connected with it by a string, were to hang over the edge of the table, the less weight would descend and draw the greater weight along the table with a velocity of 8 feet in one second; and, it is said, that if this other weight were 3*lbs.*, it would descend and draw the first equal weight of 3*lbs.* along the table with a velocity of 16 feet in one second. (*Whewell's Mechanics*, p. 239, art. 180.)

(8.) If the 1*lb.* weight suspended over the edge of the table had been left to gravitate without obstruction, its velocity would have been 16 feet instead of 8 feet in a second; hence, the resistance of the 3*lbs.* weight occasions a loss of velocity of 8 feet per second, which is equal to

the resistance of a body of equal weight moving with half its velocity in an opposite direction, or of a body half its weight, moving with an equal velocity in an opposite direction; that is, equal to half a pound weight descending by gravity; hence, the resistance of the *3lbs.* weight on the table is equal to the moving force (gravity) of a weight of $\frac{1}{2}$*lb.*; hence its resistance is $\frac{1}{6}$th of its weight, or as 1 to 6, and to the pendant weight of 1*lb.* as $\frac{1}{2}$ to 1, and as $\frac{1}{2}$ to 3, or 1 to 6 to a pendant weight of *3lbs.*; that is to say, the resistance of the *3lbs.* weight on the table against a pendant weight of *3lbs.* would be equal to that of $\frac{1}{2}$lb. moving with a velocity of 16 feet per second, which is equal to the resistance of a weight of *3lbs.* moving with a velocity of $\frac{16}{6} =$ 2 feet 8 inches per second. Hence, if these forces act simultaneously, the resulting velocity would be 16 feet — 2 feet 8 inches = 13 feet 4 inches in one second; and, consequently, according to this reasoning, if a *3lbs.* weight were suspended from the edge of the table against a *3lbs.* weight on the table, the suspended weight would descend and draw the *3lbs.* weight along the table 13 feet 4 inches in one second, not 16 feet in one second, as it is said to have been ascertained to be by experiment.

(9.) The second experiment above-mentioned would reqnire a very exact apparatus, and precise observation; certain it is, that the slightest deviation from exactitude, either in the apparatus or the observer, would make a considerable difference in the velocity of the descending weight in so short a time as one second.

(10.) Independently of which consideration, the suspended weight of *3lbs.* would (if not obstructed by the *3lbs.* weight on the table,) descend, by the force of gravity, 16 feet

in one second of time. Does this second experiment show that the suspended weight of 3*lbs.* would gravitate as fast when obstructed by the 3*lbs.* weight on the table, as if left free to gravitate without any such obstruction?

(11.) The following remark occurs in the text from which the statement of these experiments is derived (id. p. 240.) "The student might, perhaps, at first imagine that since the weight which produces velocity is *tripled*, the velocity should be three times as much as before." And the following reason is assigned for this not being the case: "In fact, however, it will only be 16 feet, or twice as much; and a little consideration will show the reason. The body which moves the other has also to move *itself* with the same velocity; and hence, the pressure arising from the first weight is not *all* employed in moving the other." These observations are applicable rather to moving forces acting independently of gravity, than to gravitating bodies.

(12.) The experiments here alluded to might have been advantageously extended to ascertain the greatest weight at the edge of the table, that the 3lbs. weight on the table would keep in equal equilibrium. (The first experiment infers that the weight would be $\frac{1}{2}$lb. because the resistance of the 3lbs. weight on the table is equivalent to $\frac{1}{2}$lb. weight acting by the impulse of gravity.)

In the following table F denotes the given moving force, and *f* the moving force which is sought, except when both are given, and then *f* denotes the moving force to which the *sought* quantity relates.

TABLE. CHAPTER IX.

Formulæ for finding the moving forces and velocities of bodies of equal weights moving uniformly.

	GIVEN.	SOUGHT.	VALUES.
1	F, V, and v	f	$f = \frac{Fv}{V}$.
2	F, f, and V	v	$v = \frac{fV}{F}$.

Formulæ for finding the moving forces and weights of bodies moving uniformly with equal velocities.

	GIVEN.	SOUGHT.	VALUES.
3	F, W, and w	f	$f = \frac{fw}{W}$.
4	F, f, and W	w	$w = \frac{fW}{F}$.

Formulæ for finding the weights and velocities of bodies impelled by equal moving forces.

	GIVEN.	SOUGHT.	VALUES.
5	W, V, and v	w	$w = \frac{WV}{v}$.
6	W, w, and V	v	$v = \frac{WV}{w}$.

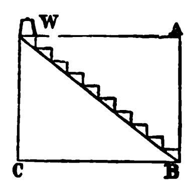
From these considerations it appears that *moving-force*, as a cause acting on a body in equilibrio, produces its equivalent effect in *the momentum* of the body in that state; so that the cause and effect being equal, the moving force is equal to the momentum. Yet, notwithstanding this equality between the two, both moving-force and momentum have each their peculiar properties which distinguish them from each other, and prevent their being altogether convertible or the converse of each other. Thus the product of the weight into the velocity is never equal to the momentum, but always exceeds it, whatever is the value of d; that is, M is always $= \frac{WV}{d}$. On the contrary, F, or the moving force, is always $=$ WV. For F has an auxiliary in that part of the force-applied which brings the body to an equilibrium, but which is not auxiliary to the momentum of the body. Hence, the moving-force is applied to the body when it is *relieved* from its weight; which relief, therefore, is in aid of the moving force; but this relief is in diminution of the momentum. In the state of equilibrium, the moving force impels the body to advantage. It moves the body possessing weight with the same facility as if it possessed no weight. Hence, when it moves the quantity of matter, it actually moves the weight; for it cannot move the one without the other. Hence in effect, with regard to moving force, $\frac{W}{d} = W$; and $F = WV$. And consequently, $W = \frac{F}{V}$; and $V = \frac{F}{W}$; which formulæ will determine all the cases of moving force with regard to any one body moving uniformly.

CHAPTER X.

OF THE COMPOSITION OF FORCES.

(ART. 1.) Impulses may be impressed on a body, either successively or simultaneously; and when impressed simultaneously they are said to be in composition.

(2.) Let an impulse be communicated to a body, W, such, that in 1″ it is moved from W to A uniformly, and there stopped, and instantly on its stopping, let another impulse be communicated to the body, such that in the next succeeding second it is moved from A to B in a direction perpendicular to WA; the body having moved with both these impulses *successively*, would at the end of 2″ arrive at the point B; that is, it would have performed the distance represented by the diagonal WB.



(3.) If these impulses were communicated successively for small parts of 1″, such as $\frac{1}{20}$″, at the end of $\frac{1}{10}$″ the body would be in the diagonal WB, and would also be found there at the end of each succeeding $\frac{1}{10}$ of a second; and if the successive times were very small, the body would

almost continually be found in the diagonal WB. But these small and minute successive impulses, by continual subdivisions, would continually approximate to a simultaneous application of both impulses to the body; and as every subdivision causes the body to be so much oftener *within* the diagonal, and when not within it, to be nearer and nearer to it, it follows, that if the impulses were simultaneous, the body would always be found in the diagonal WB, and would move from W to B in 1″. Wherefore, WB, the diagonal of the rectangle WABC, represents the *direction* of the motion, and the path of the body in the time 1″, by the composition of the two forces or impulses, and also represents the resultant velocity of the composition of the velocity, WA, and AB, of these two forces. The successive action of the forces implies a duplication of the time of simultaneous action; for,

(4.) We may remark here, that although by the subdivisions of time, the body approximates to the diagonal, yet the time will remain 2″, so long as the impulses are successive, however short and minute the intervals may be; and it is only when the forces act simultaneously that they are in composition, and then only the saving of time takes place.

(5.) These forces which are perpendicular to each other in their direction, whether applied successively or simultaneously, have no effect either to diminish or increase each other; but when applied simultaneously, then the composition of these forces changes their direction, so as to perform in the direct line WB, that distance, which if applied separately, would have been performed successively by the two lines WA and AB.

Let the velocity WA $=$ 4 rods per second;
the velocity AB $=$ 3 ditto ditto;

Then, if the forces are applied successively, the body would move 7 rods in 2″, being at an *average* rate $3\frac{1}{2}$ rods per second.

(6.) But, if the forces are applied simultaneously, the distance WB will be performed in 1″. And because the forces are perpendicular, or at right-angles to each other, WB, the diagonal, is the hypothenuse of the right-angled triangle WAB; wherefore, (by 47. 1 *Euclid*),

$$WB^2 = WA^2 + AB^2 = 4^2 + 3^2 = 25;$$

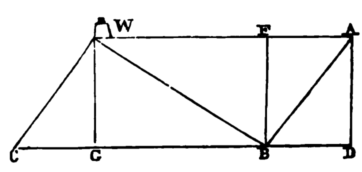
Wherefore, $WB^2 = 25$; and $WB = \sqrt{25} = 5$ rods per second; so that by this composition of the forces, WA and WB (equal together when successive to $3\frac{1}{2}$ rods per second,) their velocity is increased to 5 rods per second.

(7.) If the forces which are applied simultaneously are not perpendicular to each other, the body upon which these forces are impressed will move in the diagonal of a parallelogram, of which the lines representing the velocities of the forces in equal times, are two contiguous sides.

(8.) This proposition admits of two cases; first, where the angle of the direction of the forces is less than a right-angle; and, secondly, where it is greater.

(9.) First case, where the angle of direction is less than a right-angle.

Let the forces be represented by the right lines WA and AB, where the angle WAB is less than a right angle.



Let WAD be a right-angle, and the angle WAB being less than the right-angle WAD, then will the body impressed with the forces WA and

AB, move in the line WB, which is the diagonal of the parallelogram WABC, of which the lines WA and AB, representing the velocities of the forces, are two contiguous sides.

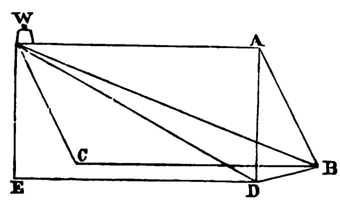
(10.) For if these forces were applied successively, the body W would be found at B at the end of the second time; but when the forces are applied simultaneously, it will (as we have seen) be always in the diagonal WB until it arrives at B, and will arrive at B in one time; e. g. 1″. Q. E. D.

Scholium.—Draw WG perpendicular to CB, and FB parallel to WG, it is obvious, that with the forces WF and WG acting at right-angles, WB would represent their resultant force and velocity; consequently, because WAB, the angle of direction of the forces is less than a right-angle, the force of WA becomes reduced to WF, and the force of AB to FB.

It is obvious, that the nearer the angle BAD approximates to a right-angle, the less will be the loss of force and velocity, and the greater its defect short of a right-angle, the greater the loss of force and velocity.

(11.) Second case: where the angle of direction is greater than a right-angle.

Let the two forces be represented by the two right lines WA and AB, and let the angle WAB be greater than the right-angle WAD; then, if the



parallelogram WABC be completed, the body impelled by these two forces simultaneously will move from W to B in the direction of the diagonal WB, in half the time that the body would move successively from W to A, and from A

to B, taking the times of the two successive motions to be equal.

Draw AD equal to AB and perpendicular to WA, and complete the parallelogram WADE, and join WD; then, if the forces WA, AB, acted at right-angles, the diagonal WD would represent the motion of the body impelled by these forces simultaneously. Join BD; and because AD (by the construction) = AB, the angle ABD = the angle ADB; but the angle ABD is greater than its part WBD; wherefore, the angle ADB is greater than the angle WBD; much more, therefore, is the angle WDB greater than the angle WBD′. But the greater side subtends the greater angle of any triangle; wherefore, WB is greater than WD, that is, the velocity of the forces WA and AB, acting at the obtuse angle WAB, is greater than the velocity would be if these forces acted at right-angles.

(12.) Hence the less obtuse the angle of direction, and the less the gain of force and velocity, and the greater the obtusity, the greater the gain.

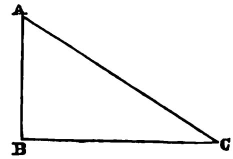
(13.) If two forces act simultaneously on a body perpendicularly to each other, the resultant velocity will be the hypothenuse of a right-angled triangle, of which the two forces WA and AD are the sides. *(See last figure.)*

For WD, the resultant velocity, is the hypothenuse of the right-angled triangle WAD, of which the two simultaneous forces WA and AD are the sides.

(14.) On the other hand, any motion or velocity whatever may be resolved into two motions produced by forces simultaneously applied to the moving body in directions perpendicular to each other.

Let ABC be a right-angled triangle, and the angle at B a right angle, and let the hypothenuse AC represent any velocity whatever; then, may the velocity AC be resolved

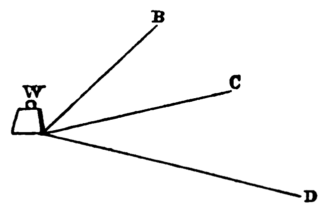
into the two velocities AB and BC, produced by two forces acting at right angles to each other.



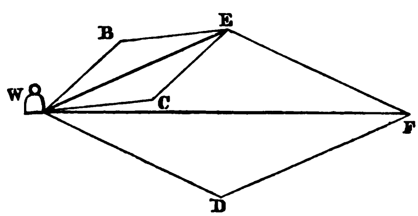
For if the velocities AB and BC had been produced by forces acting at right angles to each other, when these forces were simultaneous, the *composition* of the motion or velocities would be represented by the right line AC, which is the diagonal of the rectangle AB. BC; and, consequently, the diagonal AC may be *resolved* into the joint perpendicular actions of AB and BC from the point A; because the composition of the forces AB and BC at right angles to each other from the point A, would have produced the motion or velocity AC.

(14.) If a body be impressed with three forces in different directions, the resultant of any two of them with the third will be the resultant of all the three forces.

Let a body, W, be impressed simultaneously with three different forces, such, that if acting separately, WB, WC, and WD, would represent the motion and velocities of the body in equal times. Then will the composition of the resultant of any two velocities (that is, the resultant of WB and WC,) with WD the third force, be the resultant of all the three velocities.



If we complete the parallelogram WBEC, its diagonal WE will be the resultant velocity of the two forces WB



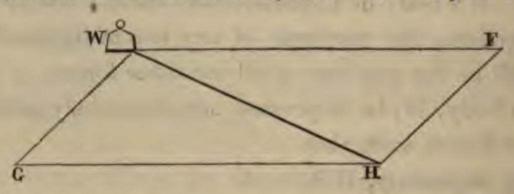
and WC; so that were there no other force impressed, the body would move in the direction of the diagonal WE, with the velocity denoted by that line.

Complete the parallelogram WEFD; the diagonal WF will be the resultant velocity of the two forces WE, WD; that is, of the composition of these two forces, and will be the resultant direction and velocity of all the three forces, WB, WC and WD.

(15.) Universally, if a body be impressed with any number of forces simultaneously, the resultant velocity of all the forces will be the composition of the resultant of all the forces but one, with that one force, in whatever order they are taken.

This is already shown as to three forces.

Besides the three forces WB, WC and WD, let a fourth



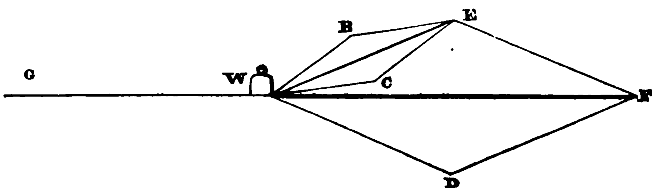
force, WG, be simultaneously impressed upon the body. Let WF be the resultant of the three forces WB, WC and WD; completing the parallelogram WFHG, the diagonal WH will represent the resultant of the direction and velocity of the forces WF and WG; that is, the resultant of the four velocities WB, WC, WD and WG.

And if a fifth force were applied, the resultant velocity of all the five forces will be found by finding the resultant velocity of four of the forces, as above; and then, by finding the resultant velocity of the fourth resultant in composition with the velocity of the fifth force; and so of any greater number of forces.

(16.) If there be any number of forces acting simul-

taneously upon a body, and if the velocity produced by one of these forces is equal to the resultant velocity of all the others, and acts in an opposite direction to it, the body will remain at rest.

Let the diagonal WF represent the resultant velocity of the three forces WB, WC and WD, being three forces



which, if applied simultaneously, would carry the weight from W to F; and let a fourth force be also applied simultaneously with the three others in the direction of WG opposite to that of the resultant WF, and equal to it.

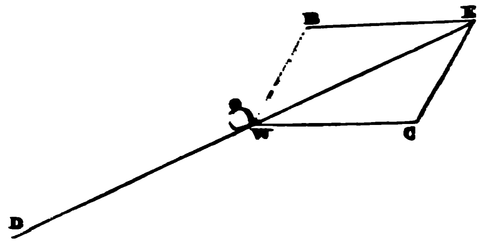
Because WF is equal to WG, and these forces act simultaneously upon the body in direct opposition to each other, their simultaneous application would occasion the body to remain at rest. (Art. 2, chap. vi.)

(17.) If any three forces, acting upon a body simultaneously, keep it in a state of rest, these three forces, if applied successively, would cause the body to describe a triangle, whose sides would represent each force as applied in succession, and its direction; the first force commencing at the original position of the body, and the third velocity terminating there.

If any three forces, acting upon the body W (see diagram next page), simultaneously, keep it in a state of rest, the diagonal representing the composition of two of the velocities is equal to, and in a contrary direction to, the right line representing the third velocity.

Let one force produce the velocity represented by the

right line WB, another force the velocity BE: the diagonal WE represents the composition of these forces acting in the direction from W to E.



Wherefore, a third velocity, acting simultaneously with the others, that would keep the body in a state of rest at W, must be equal and opposite to the direction of the diagonal WE. (art. 2. chap. vi.) Let WD be = WE, the velocity WD being equal and in an opposite direction to WE, the composition of the other two velocities would counteract them, and leave the body at rest by this simultaneous action of the forces.

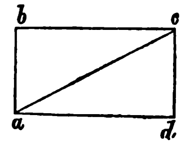
But if the forces acted successively, they would describe the triangle WBE, whose sides WB, BE and EW would represent the velocity of each force as applied in succession, and its direction; the first velocity WB commencing at W, the original position of the body, and the third velocity, equal to WD, ending there.

(18.) *Corollary* 1. If any two sides of a triangle represent two forces acting upon a body, the third side will represent the resultant of the other two forces.

(19.) *Coroll.* 2. If three forces represented by the sides of a triangle act upon a body simultaneously, they will keep the body in equilibrio, taking the directions of the forces, to be such as they are described in article 17.

(20.) If any two forces act at right angles to each other, their simultaneous action will neither diminish nor increase each other.

Let *ab* and *ad* represent any two forces, acting at right angles to each other; the simultaneous action of these two forces *ab*, and *ad*, will neither diminish nor increase each other.



The parallels *ab* and *cd*, being equidistant at the opposite points *b* and *c*, and *a* and *d*; if the force *ad* acting at right angles to *ab*, were diminished by the joint action of the force *ab*, the simultaneous action of these forces *ab* and *ad* would occasion the body moved to arrive at some position, *c*, where the distance *bc* would be less than the distance *ad*. But the diagonal *ac* represents the distance actually performed by the simultaneous action of the two forces *ab*, and *ad*. (art. 2.)

And because *abcd* is a parallelogram, the side or distance *bc* is equal to the side or distance *ad*: that is, the distance *bc* is not less than *ad*; wherefore the force *ab* does not diminish the force *ad*.

And in the same manner it may be demonstrated, that the force *ad* acting perpendicularly to the force *ab* does not diminish it: for *dc* is = *ab*.

Again these perpendicular forces do not increase each other; for in that case *bc* would be greater than *ad*, but *bc* is = *ad*.

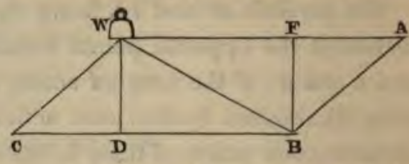
Wherefore if any two forces, &c. Q. E. D.

(21.) *Coroll.* Forces which act at a right angle to each other are neither opposing nor auxiliary, but neutral, forces.

(22.) If two unequal forces are simultaneously impressed at an angle less than a right angle, then will the *action* of the less force be to the less force as the perpendicular to the hypothenuse of a right-angled triangle, of which the right line representing the less force is the hypothenuse, and the perpendicular is a right line drawn from the extremity of the hypothenuse at right angles to the right line represent-

ing the greater force; and the *action* of the greater force will be the abscissa of the right line representing the greater force cut off by the perpendicular.

Let the right lines WA and AB represent the two forces acting from W to A, and from A to B, at the angle WAB, which is less than a right angle; and let the perpendicular BF be drawn from B to WA. Then will the base BF represent the *action of the less force* AB; and the *action of the greater force* WA will be represented by the abscissa WF.



Complete the parallelogram WABC, and draw WD equal and parallel to FB. Because the forces WA and AB act simultaneously at W, the body will move in the diagonal WB, and by article 13, their joint action may be resolved into the joint action of the two forces WF and FB; wherefore the action of the force WA is equal to the force WF; and the action of the force AB is equal to the force FB; but FA is the base of the right-angled triangle, of which AB, representing the less force, is the hypothenuse, and FB is the perpendicular; and WF, which represents the action of the greater force WA, is the abscissa of WA cut off by the perpendicular FB.

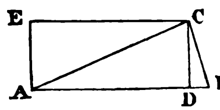
Wherefore if two unequal forces, &c. Q. E. D.

(23.) *Coroll.* Forces which act simultaneously at less than a right angle to each other, are opposing forces.

(24.) If two equal forces act upon a body at an angle less than a right angle, the *action* of one force in opposition to the other force is as the base to the hypothenuse of a right-angle triangle, of which the perpendicular shall be a right line drawn from the end of the right line, representing

the opposing force at a right angle to the right line representing the force opposed; and of which right-angled triangle the right line, representing the opposing force, shall be the hypothenuse.

Let the two equal right lines BA and AC represent two equal forces impressed upon a body at B, and acting in the directions BA and AC, at the angle BAC, less than a right angle. If the forces



were impressed successively at B, the body would move from B to A, and then from A to C; but when impressed simultaneously, the body would move from B to C in the diagonal BC. Draw CD perpendicular to AB. Then will the opposing *action* of the force AC be to the *force* AB as the base AD to the hypothenuse AC.

For BC is also the resultant of the two forces CD, and DB: and DB is part of AB; wherefore the action of the force AC diminishes the force AB to the force DB, and consequently the opposing force AC is = AB — DB = AD.

But AD is the base, and AC the perpendicular of the right-angled triangle ACD, of which AC is the hypothenuse, and the perpendicular CD is the right line drawn from the end of AC, representing the opposing force, at right angles to AB, which represents the force opposed.

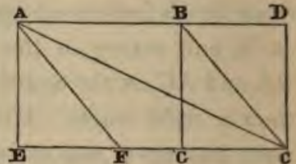
Wherefore, if two equal forces, &c. Q. E. D.

(25.) *Coroll.* Since (by the third law of motion) the *reaction* of AB would be equal to AD the action of AC, the abscissa DB would represent that part of each force which would *not act* in opposition to the other.

(26.) If two equal forces act together simultaneously at an angle greater than a right angle, then will the *increase* of the *action of one force* be to the other *force* as the base is to the hypothenuse of a right-angled triangle of which the

right line representing one force is the hypothenuse, and of which a right line drawn from the extremity of the hypothenuse, perpendicular to the right line produced, which *represents* the other force is the perpendicular.

Let the equal right-lines AB and AF represent two equal forces acting simultaneously at the point A. The diagonal AC will represent their joint action. Produce AB towards D, and draw CD perpendicular upon AD; and draw BG perpendicular to FC; producc CF towards E; and draw AE perpendicular to EC.



BD denotes the *increase* of the action of the force AB gained by the angle ABC (which is the angle of the direction of the forces) being greater than EAB, which is a right angle.

But BD is to BC, that is, to AF the other force, as the base is to the hypothenuse of the right-angled triangle BDC, of which the right line BC, representing one force, is the hypothenuse, and DC = BG, drawn from the extremity of the hypothenuse BC, perpendicular to AB produced, is the perpendicular.

Wherefore, if two equal forces act together, &c. Q. E. D.

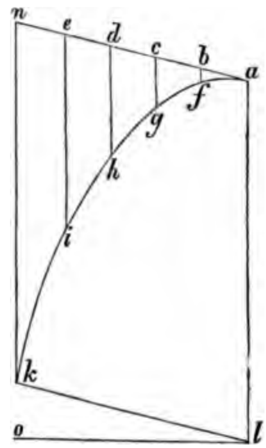
(27.) *Corollary.* Equal forces, which act simultaneously on a body at an angle greater than a right angle, are auxiliary to each other.

Problem.

(28.) To find the path described by a body, projected at an angle less than a right angle with the horizon.

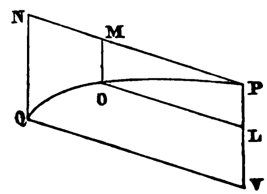
Let the body be projected with an uniform velocity from *a* towards *n*, which, acting alone, would describe the equal

spaces *ab, bc, cd, de,* and *en,* in equal times, and so that in the time in which it would describe *ab,* it would descend by gravity through the space *bf*; the body, by the simultaneous action of the projecting force, and of gravity, would at the end of that time be in the point *f*; and because the spaces described by gravity are as the squares of the times, then, when the body would at the end of two times have described the space *ac,* it would have descended through $2^2 = 4$ times the space it descended through in the first time. Describe $cg = 4bf$; then, by the joint action of both forces, the body would be at the point *g* at the end of the second time. In like manner describe $dh = 3^2 =$ 9 times *bf,* and $ei = 4^2 = 16$ times *bf,* and $nk = 5^2 = 25$ times *bf,* the body will be in the points *h. i* and *k,* at the end of the third, fourth, and fifth times. And by taking fractions of these times the intermediate situations of the body would be found to be in the curved path *afghik.* Which was to be done.



Note.—Mr. Bridge seems to *assume that the path of the body will be a curve;* and it does not very clearly appear whether its being a curve is the foundation of his subsequent reasoning, or his reasoning is intended to demonstrate that the path is a curve. We have, therefore, considered it more intelligible to the reader, first to describe the path (in the above problem), and then to demonstrate (in the following theorem) that the path is a part of a parabola. (See *Bridge's Mechanics,* p. 52, part i.) Unfortunately the diagram to which he refers in his reasoning is not a happy illustration of it.

(*See margin.*) The parallelogram PMOL, is *equiangular* with the parallelogram PNQV; and a beginner would be apt to mistake them as being *similar* parallelograms; which, if they were, would altogether disprove Mr. Bridge's reasoning. (See scholium to the next theorem.) In a word, by his diagram the eye is not made to assist the reasoning faculty (which is the great use of diagrams), but rather to perplex and obstruct it.

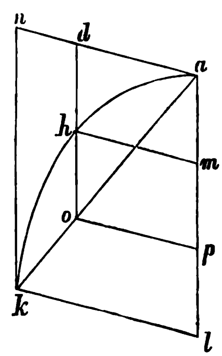


Theorem.

(29.) The path described by a body projected at an angle less than a right angle with the horizon is a parabola. (*See the figure in p.* 105.)

Let any two points h and k be taken in the path described by the body, by the construction $nk : dh :: na^2 : da^2$.

By coroll 2. to prop. 12, *Rt. Simson's Conic Sections*, and by *Bridge's Conic Sections*, prop. 9. lecture 2., the squares of the ordinates are as the abscissas; that is, the square of the ordinate kl, is to the square of the ordinate hm, as the abscissa la is to the abscissa ma.



But because of the parallels, the ordinate $kl = na$, and the ordinate $hm = da$, and the abscissa $la = nk$, and the abscissa $ma = dh$; hence, by the properties of the parabola, $nk : dh :: na^2 : da^2$.

And the same will be the case whatever point we take in the path of the projectile or projected body; wherefore, the path described by a body projected, &c. Q. E. D.

Scholium.

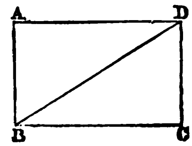
(30.) We may here remark, that if the force of gravity generated an uniform velocity (instead of a velocity uniformly accelerated,) so as uniformly to describe *nk* or *al* in the same time that by the projecting force the body would describe *an*; then, at the end of the time *ad* the body would be at the point *o* instead of the point *h*; for the body would move in the diagonal *ak* in that case; and, therefore, would always be in a point, such that *do* would be to *nk* as *da* to *na*; because the parallelograms about the diagonal *ak* are similar (6 *Euclid*, 24.), and, therefore, their sides are proportionals (*Euclid* 6. *Definition* 1.); and, consequently, *do* to *nk*, would be as *da* to *na*. Hence, though the parallelograms *dm*, and *nl* are equiangular, they are not similar, because their sides about their equal angles are not proportionals, but the parallelograms *dp* and *nl* are both *equiangular* and *similar*. The reader cannot be too strongly impressed with this distinction.

(31.) We have thus shortly considered the nature of the composition of forces, principally for the purpose of demonstrating the power of the inclined plane, and the application of the lever, or wheel and axle, obliquely to the direction of the spiral of the screw. The following examples will serve to show in what manner some of the properties of compound forces are brought into use.

Example 1.

(32.) If two forces act jointly upon a body at B, at right angles to each other, one of which would move the body 6 feet in the same time in which the other force would move the body 8 feet: What space will the body move over by the simultaneous action of both forces?

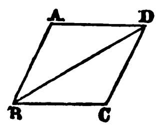
Let AB = 6 feet. and BC = AD (perpendicular to AB) = 8 feet; and draw the diagonal BD. By art. 2, the body would move from B to D by the joint action of both forces, and would describe the space BD. But BD is the hypothenuse of the right-angled triangle ABD; wherefore, (1 *Euclid*, 47,) $BD^2 = AB^2 + BC^2 = 6^2 + 8^2 = 100$; and extracting the square root on both sides $AB = \sqrt{100} = 10$ feet; the answer.



Example 2.

(33.) If any two forces, BA, BC, act upon a body at B in the directions BA. BC, at an angle of 60° to each other; What is their compound force?

BD, the diagonal of the parallelogram ABCD, represents this compound force.



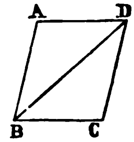
Because AD is parallel to BC, the two interior angles DAB, ABC, are together equal to two right angles = 180° (1 *Euclid*, 29.); wherefore the angle DAB is = 180 − 60 = 120°, and the equalangles ABD. ADB, are each = 30°. Wherefore (by proposition 7, *New Supplement to Euclid*)

$$BD^2 = 3 \times AB^2;\text{ and extracting the square root,}$$

$$BD = \sqrt{(3 \times AB^2)};\text{ the answer.}$$

Example 3.

(34.) If two unequal forces, BA, BC, act upon a body at B in the directions of the given angle ABC: What will be the given ratio of their compound force?



Because the angle ABC is given, the angle BAD is given = (180° − angle ABC). Hence two sides BA, AD, and the included angle BAD are given of the triangle BAD, and the side BD is sought.

By formulæ 3, in the table, for cases of oblique-angled triangles (*Simson's Trigonometry*, appended to *Euclid*), if two sides and the included angle are given, the two other angles ADB and ABD may be found; and by formula 1, in that table, if the three angles are given or found, and one of the sides AB is given, the remaining sides BD and AD may be found; that is, sine ADB : sin. BAD : : AB : BD. And because the sines of the known angles ADB, BAD, may be found by the trigonometrical tables, and AB is given; three of the four proportionals,

$$\text{sin. ADB} : \text{sin. BAD} :: \text{AB} : \text{BD},$$

are given, or known quantities; whence BD may be found by the Rule of Three direct; from which Rule it follows, that

$$\text{BD} = \frac{(\text{sin. BAD}) \times \text{AB}}{\text{sin. ADB}}; \text{ the answer.}$$

END OF PART I.

PART II.

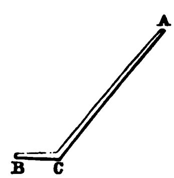
THE MECHANICAL POWERS.

CHAPTER I.

THE LEVER.

(Art. 1.) Mechanics is the science of the application of power or force to move bodies. The principal instruments or agents employed in this application of force are of six kinds:—1. The lever; 2. the wheel and axle; 3. the pulley; 4. the inclined plane; 5. the wedge; and 6. the screw. When any of these instruments are combined with one or more of the others, the combined instrument is called a *machine*. The force which is applied by means of these instruments or machines is usually called in mechanics *the power;* and the resistance of the body to be moved, *the weight*.

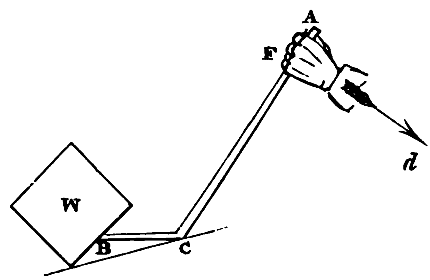
Fig. 1.



(2.) The instrument called the lever is an inflexible bar or rod, generally of iron, sometimes straight, but generally bent near one end, as in fig. 1. The bent lever is usually called a crow or crow-bar.

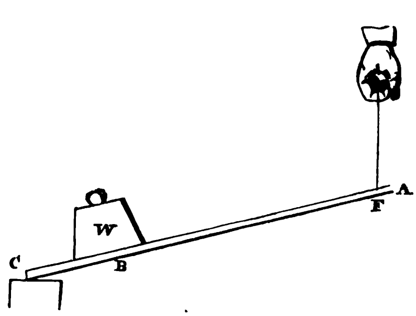
(3.) The crow, or bent lever, is used in raising or moving heavy bodies, in the following manner; its end B (fig. 2.) is applied to the body W, so that the point of its angle C, may touch some hard and immoveable substance, such as rock, &c., which thus becomes the *fulcrum.* The operator then pulls the lever at or near the other end A, in the direction of *d,* the arrow in fig. 2, opposite to the direction in which he wishes to move the body W.

Fig. 2.



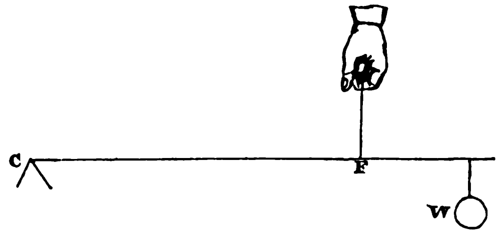
(4.) The straight lever is used as follows: the one end C, of the lever is placed against the fulcrum, being some hard and immovable substance, and under the body W, as in fig. 3; the force F is applied at or near the other end A so as to raise the body W. In this case one end of the lever is at C, the fulcrum, and the body or weight is between the power and the fulcrum. In the case of the bent lever the fulcrum is between the body and the power.

Fig. 3.



(5.) By both these kinds of levers power is acquired or

increased. There is a third kind by which the power is diminished. This third kind is also a straight lever,



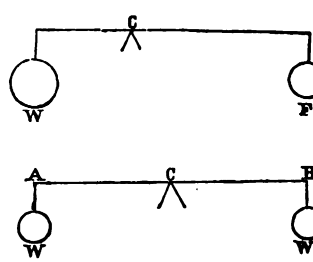
where the power or force acts between the weight W and the fulcrum C.

Section I.

Of the Lever, when the Fulcrum is between the Power and the Weight.

(6.) The first kind of lever may be more conveniently represented by considering the power as a weight suspended from that part of the lever to which the power or force is applied; in which case the straight lever may be substituted for the bent one, as in the margin.

(7.) If a straight iron bar is nicely balanced on the point of a fulcrum under the middle of it, so that the length of the arm AC is equal to that of the arm CB; and if two equal weights be suspended from the ends A and B of each arm, they will balance each other, and remain at



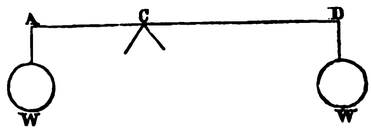
rest in equilibrium. For the direction of the action of the weights is perpendicularly downward, and the descent of

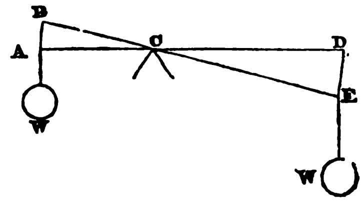
one would cause the other to rise as far as the first one fell contrariwise to the original direction of the action of the ascending weight; and their velocities are equal, because they would describe equal arcs of the same circle in the same time, and because the weights and velocities are equal, their products would be equal, that is, $WV = wv$; and because $F = WV$ and $f = wv$, $F = f$, that is, their forces would be equal (theor. 1, chap. ix.); and the action of one would generate an equal reaction in the other, which would resist it. And the resistance being equal to the force, mutually between the weights, neither force could prevail over the other, and the weights must remain suspended at rest in equilibrium, from the joint operation of these equal, opposite, and balancing forces.

(8.) But if one of the weights were heavier than the other, the heavier weight would descend, and the lighter ascend, with a velocity proportioned to the *excess* of the heavier weight above the lighter. *This excess constitutes the moving force* (as we have seen in chap. ix. art. 3, part i.) Let the heavier weight = 4*lbs.* and the lighter weight 3*lbs.* the *moving* force will not be 4*lbs.* but $4 - 3 = 1$*lb.* For the action of the heavier weight produces a reaction from the lighter weight, equal (not, as is too often erroneously imagined, to the action of the heavier weight, for then the weights would not move at all, but) to the action of the lighter weight = 3*lbs.*, which balances an equal quantum of the force of the greater weight, leaving only the excess of weight, 1*lb.*, as the moving force. Thus, if the power of one horse applied to a given draught be = 2 tons, and the weight to be drawn be = 1 ton, the *moving* force of the horse would be only 1 ton; and if the weight were drawn by two horses of equal power with the first, equal together to 4 tons, the moving force of the two horses would be $4 - 1 = 3$ tons; that is, the moving

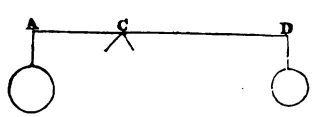
force of the two horses drawing the weight would be treble the moving force of one horse. So the power of three such horses would be $2 \times 3 = 6$ tons; but if applied to this weight the moving force would be $6 - 1 = 5$ tons; the power of 4 horses $= 2 \times 4 = 8$ tons; their moving force $= 8 - 1 = 7$ tons; and so on; so that the moving forces of the horses would increase as the odd numbers, 1, 3, 5, 7, 9, &c. And because the velocities of the successive number of horses would be as their moving forces,(theor. 1, chap. x. part 1.), these velocities being uniform, the spaces described are as the velocities (gen. theorem, chap. iii.); that is, as the odd numbers, 1, 3, 5, 7, 9, &c., which are precisely analogous to the ratios of the spaces described by a gravitating body in equal successive times. Hence, if we are not digressing too far, we may reasonably conclude that gravity is a force, which, when applied, would without resistance describe a space of 32 feet in the first second, 64 feet (that is, double) in the second second, 96 feet, treble, in the third second, and so on in an arithmetical progression; but that this original force of gravity is resisted by the reaction or counteraction of the gravitating body; that this resisting reaction diminishes the spaces which would otherwise be described by the applied force of gravity, by an uniform retardation of 16 feet per second; that the spaces actually described are the effect of the excess of the applied force of gravity above the reaction of the gravitating body, and that this excess constitutes the moving force of gravity.

(9.) If the bar or rod is not supported by a fulcrum exactly midway between the two equal weights, but nearer to one than the other, the further weight will preponderate, and the end of the arm CD will descend with an uniform velocity in a certain time from D to E; and in the same time the end of the arm CB will ascend from A to B. In





this case the force of the weight is applied at D, with an *advantage* over the equal weight at A in point of *position.* This advantage is the *leverage:* by means of which the action of the weight at D is greater than the resisting re-action of the weight at A. This leverage or advantage is the moving force of the weight at D. And because the space DE is described uniformly, it represents the velocity of the motion of the weight from D to E. And (chap. ix. page 91.) the moving force is equal to the product of the weight into the velocity; wherefore, F $=$ W $\times$ DE. Hence, the cause of this advantage is, that the weight at D moves with a greater velocity than the weight at A. For DE and AB, which represent the velocities, are arcs of two concentric circles, of which C is the common centre; and the vertical angles DCE and ABC are equal to each other (1 *Euclid,* 15.) But arcs of circles which subtend equal angles at the centre are as the radii of the circles (proposition 21, *New Supplement to Euclid*); wherefore, CD : AC : : DE : AB, that is, since CD is greater than AC : DE, the velocity at D will exceed AB, the velocity at A, in the same ratio.



(10.) In order to balance the weight at A, a lighter weight should be suspended at D, whose action would be equal to the reaction of the weight at A. For the action of the weight at D will produce a greater velocity in that weight than in the opposite one at A. Now in order to

create an equilibrium, WV, and wv must be equal to each other; but when $WV = wv$, then reducing these products into the form of proportionals, $W : w :: v : V$.

Thus, if AC be $= 1$ foot, and CD $= 2$ feet, then because $AC : CD :: AB : DE$ (*see the figure*, art. 9.), the velocity (V) of the weight A will be to v, the velocity of the weight D, as 1 to 2. But the weights are in the inverse ratio of the velocities, that is, 2 to 1; hence, the weight required at D to sustain in equilibrium 100*lbs.* weight at A would be 50*lbs.*, for $2 : 1 :: 100 : 50$; and, consequently, by means of a lever of these proportions, the power becomes doubled in its effect. Hence, if it were possible to increase the leverage infinitely, the power might be infinitely increased also, which amounts to what some have considered the vain boast of Archimedes,—

"Δος που στω, και κινησω τον κοσμον *."

Let P represent the power applied equal to one of the weights W;
L = CD, the longer arm of the lever;
w the weight;
and l = AC, the shorter arm of the lever.

Theorem 1.

If the power and weight be in equilibrio, then will the product of the power into the longer arm be equal to the product of the weight into the shorter arm; that is $PL = wl$.

* "Give me a place to stand upon, and I will displace the world." Some (who translate που στω, a fulcrum,) think he would have used a lever; but I question whether he would not have used pulleys, as being more handy.

For, as we have seen in art. 9, (*see the second figure in* art. 9,) CD : AC : : DE : AB; and because L = CD, and l = AC; L : l : : DE : AB; but DE represents the velocity, V, of L, and AB represents the velocity, v, of l.

Wherefore,	$L : l :: V : v$;
And conversely,	$l : L :: v : V$;
But (by art. 10.) when the weights are in equilibrio	$W : w :: v : V$;
Wherefore, (*ex equali*)	$W : w :: l : L$;
And multiplying the extremes and means	$WL = wl$.

But w represents the weight, suspended at the end of the short arm l, or AC; and W, the weight suspended at the end of the long arm L, or CD; and if this latter weight W be considered the power P, then substituting P for W,

$$PL = wl.$$

Theorem 2.

If the power and weight be in equilibrio, the power will be to the weight, as the shorter arm to the longer arm; that is, $P : w :: l : L$.

By theorem 1, $PL = wl$; and (transforming these products of extremes and means into proportionals),

$$P : w :: l : L.$$

Theorem 3.

If the power and weight be in equilibrio, the power is equal to the product of the weight into the shorter arm of the lever, divided by the longer arm; that is,

$$P = \frac{wl}{L}.$$

For by theorem 1, $PL = wl$; and dividing each of these equals by L,

$$P = \frac{wl}{L}.$$

Theorem 4.

If the power and weight be in equilibrio, the weight is equal to the product of the power into the longer arm of the lever, divided by the shorter arm; that is,

$$w = \frac{PL}{l}.$$

By theorem 1, $PL = wl$; and dividing by l,

$$\frac{PL}{l} = w.$$

Theorem 5.

If the power and weight be in equilibrio, the longer arm of the lever is equal to the product of the weight into the shorter arm, divided by the power; that is,

$$L = \frac{wl}{P}.$$

By theorem 1, $PL = wl$; and dividing by P,

$$L = \frac{wl}{P}.$$

Theorem 6.

If the power and weight are in equilibrio, the shorter arm of the lever is equal to the product of the power into the longer arm, divided by the weight, that is, $l = \frac{PL}{w}$.

By theorem 1, $PL = wl$; and dividing by w,

$$\frac{PL}{w} = l.$$

We may simplify these formulæ, by taking l, the shorter arm as $= 1$, and L, the longer arm, equal to a number having the same ratio to 1, which the length of the longer arm bears to that of the shorter; and we may substitute this ratio for the original ratio of l to L (by prop. 18, chap. ii. *New Introduction*). For if the length of l expressed in inches be 12 inches, and the length of L, also expressed in inches, be 72 inches, these lengths expressed in feet would be $l = 1$ foot, and $L = 6$ feet, which are as 1 to 6; being the same ratio as that of 12 to 72. Also if $l = 10$ inches and $L = 60$ inches; then if l be taken as $= 1$, that is, $= \frac{10}{10}$, L may be taken as $= \frac{60}{10} = 6$; for $10 : 60 :: 1 : 6$.

By this substitution we shall be able to exterminate l from our formulæ; for if we make $l = 1$, and by transposition, make it the first term of the four proportionals, then (by prop. 11, chap. xi. *New Introduction*) the fourth term will be equal to the product of the means, as will appear from the following theorems.

When the power and weight are in equilibrio.

Theorem 7.

If l (whatever be its value) be taken as equal to 1, and L be taken in a similar ratio to its value, then will the power be equal to the quotient of the weight, divided by the longer arm; that is,

$$P = \frac{w}{L}.$$

By theorem 2, $P : w :: l : L$; and by transposition,

$l : L :: P : w$; and because the products of the extremes and means are equal,

$$lw = LP;$$

and because (by the hypothesis) $l = 1$, $lw = w$,

$$\therefore w = \text{LP};$$

and dividing these equals by L, $\frac{w}{\text{L}} = \text{P}$.

Theorem 8.

If l be taken $= 1$, the weight will be equal to the product of the power into the longer arm; that is, $w = \text{PL}$.

By theorem 7, $w = \text{LP}$. (Being the last step but one in the demonstration.)

Theorem 9.

If l be taken $= 1$, the longer arm will be equal to the weight divided by the power; that is,

$$\text{L} = \frac{w}{\text{P}}.$$

By theorem 8, $w = \text{LP}$; and dividing each of these equals by P,

$$\frac{w}{\text{P}} = \text{L}.$$

From these last three theorems 7, 8, and 9, we collect the following formulæ;

$$\text{P} = \frac{w}{\text{L}};\ w = \text{PL};\ \text{and}\ \text{L} = \frac{w}{\text{P}};$$

From which, if any two of the three quantities, P, W, and L, be given, the third may be found, as will appear in the following table.

TABLE I. PART II. CHAPTER I. § 1.

A Table of Formulæ for finding the power, weight and leverage constituting an equilibrium by means of a lever, whose fulcrum is between the weight and the power, taking the shorter arm of the lever as equal to 1.

	GIVEN.	SOUGHT.	VALUES.
1	w and L	P	$P = \frac{w}{L}$
2	P and L	w	$w = PL$
3	w and P	L	$L = \frac{w}{P}$

Problem 1.

Given the weight and the long arm of the lever: it is required to find the power requisite to produce an equilibrium.

EXAMPLE.

With a lever, whose longer arm is 8 feet, and whose shorter arm is 1 foot, what power will be required to keep in equilibrium a body weighing 2 tons?

Here w is given = 2 tons = 40 cwt., and L is given = 8, and P is sought.

By formula 1, $P = \frac{w}{L} = \frac{40}{8} = 5$ cwt. the power required.

Problem 2.

What weight will a power of 4*lbs.* applied at the end of the long arm of a lever, the length of which arm is 6 feet, and the length of the shorter 1 foot, keep in equilibrium?

Here P is given = 4 *lbs.*, and L is given = 6 feet; and w is sought.

By formula 2, $w = \text{PL} = 4 \times 6 = 24$*lbs.*; the weight required.

Problem 3.

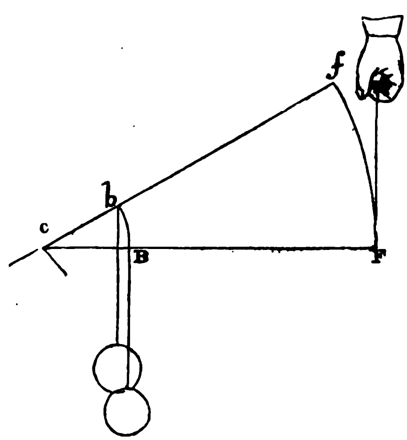
A stone of $1\frac{1}{2}$ ton weight is to be raised by two men, whose power together is equal to 6 cwt. : what leverage will they require to keep the stone in equilibrium?

Here w is given = $1\frac{1}{2}$ ton = 30 cwt.; and P is given = 6 cwt., and L is sought.

By formula 3, $\text{L} = \frac{w}{\text{P}} = \frac{30}{6} = 5$; that is, the length of the longer arm must be as 5 to 1 to that of the shorter arm.

Section 2.

Of the second kind of Lever, where the weight is between the Power and the Fulcrum.



Let C be the fulcrum, and B the point of the lever from which the weight w is suspended. The arc Ff will represent the velocity of the power at F; and the arc Bb, the velocity of the weight at B in raising the weight from B to b. Because arcs of circles which subtend equal angles at the

centre are as the radii of the circles (prop. 1, *New Supplement to Euclid*),

$$CF : CB :: Ff : Bb.$$

Let P represent the power equal to W the weight applied at F,

L, the whole lever CF;

w, the weight at B;

and l, the abscissa, or the part CB of the lever between the fulcrum and the weight.

Theorem 1.

If the power and weight be in equilibrio, then will the product of the power into the *whole lever* be equal to the product of the weight into the abscissa of the lever, between the weight and the fulcrum; that is, $PL = wl$.

By art. 10, $WV = wv$;

But W is $= P$; wherefore, $PV = wv$; and $V : v :: L : l$ (theorem 1. p. 116.); and substituting L and l, for V and v, and P for W; $PL = wl$.

Theorem 2.

If the power and weight be in equilibrio, the power will be to the weight as the abscissa to the whole lever; that is,

$$P : w :: l : L.$$

By theorem 1, $PL = wl$; and (resolving these products of the extremes and means into their proportional form),

$$P : w :: l : L.$$

Theorem 3.

If the power and weight be in equilibrio, the power is equal to the product of the weight into the abscissa, divided by the length of the whole lever; that is,

$$P = \frac{wl}{L}.$$

By theorem 1, $PL = wl$; and dividing each of these equals by L,

$$P = \frac{wl}{L}.$$

Theorem 4.

If the power and weight be in equilibrio, the weight is equal to the product of the power into the whole length of the lever, divided by the abscissa; that is,

$$w = \frac{PL}{l}.$$

By theorem 1, $PL = wl$; and dividing by l,

$$\frac{PL}{l} = w.$$

Theorem 5.

If the power and weight be in equilibrio, the length of the whole straight lever is equal to the product of the weight into the abscissa, divided by the power;

that is, $L = \frac{wl}{P}$.

By theorem 1, $PL = wl$; and dividing by P,

$$L = \frac{wl}{P}.$$

Theorem 6.

If the power and weight are in equilibrio, the abscissa is equal to the product of the power into the length of the straight lever, divided by the weight; that is,

$$l = \frac{PL}{w}.$$

By theorem 1, $PL = wl$; and dividing by w,

$$\frac{PL}{w} = l.$$

From the last four theorems we collect the following formulæ:

$$P = \frac{wl}{L}; \text{ by theorem 3.}$$

$$w = \frac{PL}{l}; \text{ by theorem 4.}$$

$$L = \frac{wl}{P}; \text{ by theorem 5.}$$

$$l = \frac{PL}{w}; \text{ by theorem 6.}$$

If any three of the four quantities P, L, w, and l of the lever, where the weight is between the fulcrum and the power, are given, the remaining quantity may be found by the following table:

TABLE II. PART II. CHAPTER I. §. 2.

A Table of Formulæ for finding the power, weight, and leverage, constituting an equilibrium by means of a lever, where the weight is between the power and the fulcrum.

	GIVEN.	SOUGHT.	VALUES.
1	w, l and L	P	$P = \frac{wl}{L}$
2	P, L and l	w	$w = \frac{PL}{l}$
3	w, l and P	L	$L = \frac{wl}{P}$
4	P, L and w	l	$l = \frac{PL}{w}$

Problem 1.

Required the power that would keep in equilibrio a lever, where the weight is between the power and the fulcrum, on which a weight of 2 tons is placed at the distance of 7 feet from the power, and 1 foot from the fulcrum?

Here w is given $=$ 2 tons $=$ 40 cwt., and L is given $=$ $7 + 1 = 8$ feet, and l, the abscissa, is given $= 1$ foot, and P is sought.

By formula 1, $P = \frac{wl}{L} = \frac{40 \times 1}{8} = 5$ cwt.; the answer.

Problem 2.

What weight will a power of 2 cwt. keep in equilibrio when applied at one end of a lever of this kind, upon which the weight is placed at the distance from the fulcrum of $\frac{1}{9}$ the length of the lever?

Here P is given $=$ 2 cwt., L is given $= 9 + 1 = 10$; l is given $= 1$, and w, the weight is sought.

By formula 2, $w = \frac{PL}{l} = \frac{2 \times 10}{1} = 20$ cwt. $=$ 1 ton; the answer.

Problem 3.

It is required to find the length of a lever which, with a power of 5 cwt. at one end, will keep in equilibrio a weight of 6 tons placed between the fulcrum and the power, at the distance of 6 inches from the fulcrum.

Here w is given $=$ 6 tons $=$ 120 cwt., and l is given $=$ 6 inches, and P is given $=$ 5 cwt., and L is sought.

By formula 3,

$L = \frac{wl}{P} = \frac{120 \times 6}{5} = \frac{720}{5} = 144$ inches $=$ 12 feet; the answer.

Problem 4.

With a lever 8 feet long, at what distance from the fulcrum, between the fulcrum and the power, must a weight of 2 tons be placed, to keep in equilibrio a power of 4 cwt. acting in an opposite direction at the other end?

Here P is given = 4 cwt., w is given = 2 tons = 40 cwt., and L is given = 8 feet = 96 inches, and l is sought.

By formula 4,

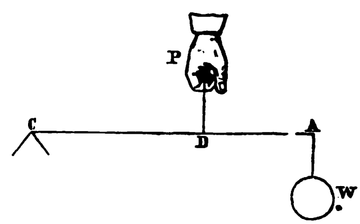
$$l = \frac{PL}{w} = \frac{4 \times 96}{40} = \frac{384}{40} = 9\tfrac{3}{5} \text{ inches};$$

the length of the abscissa; the answer.

Section 3.

Of the Lever, when the Power is between the Fulcrum and the Weight.

Since power is lost by a lever of this kind, it is seldom used but when the necessity of the case requires it. It may happen occasionally, from some peculiarity of the position of the weight, that the power cannot be applied otherwise than to this disadvantage. We shall notice the principal properties of this enfeebling instrument, without going into detail.



Let AC be the lever, W the weight suspended at A, C the fulcrum, and P the power applied at D. It is obvious, that if the weight descended by the force of gravity, the arc described from

the point A would be greater than the arc described from the point D, and these arcs represent the velocities of forces acting at A and D; let the power be represented by an equivalent weight w; and because the forces are equal to the products of the weights into the velocities, and the force at A (F) $=$ WV; and the force at D (f) $= wv$; and when these forces are in equilibrium $\mathrm{WV} = wv$; that is, by resolving this equation into proportionals,

$$\mathrm{W} : w :: v : \mathrm{V}.$$

But because the arcs at A and D, that is, V and v, are as the radii AC, and DC, that is as L and l; by substituting l and L for v and V, $\mathrm{W} : w :: l : \mathrm{L}$; but w is the power (P); wherefore,

$$\mathrm{W} : \mathrm{P} :: l : \mathrm{L};$$

that is, the weight is to the power as the abscissa or short of the lever CD is to the whole lever AC.

From this expression we obtain the following formulæ:

$$\mathrm{WL} = \mathrm{P}l; \text{ hence, } \mathrm{P} = \frac{\mathrm{WL}}{l}; \; l = \frac{\mathrm{WL}}{\mathrm{P}}; \; \mathrm{W} = \frac{\mathrm{P}l}{\mathrm{L}};$$

$$\text{and } \mathrm{L} = \frac{\mathrm{P}l}{\mathrm{W}}.$$

From which formulæ, if any three of the four quantities W, P, L, and l, be given, the fourth may be found.

In this chapter we have considered the lever as an instrument *without weight*, which (the weight) in general may be disregarded, although the weight of the lever itself when employed in raising weights has some effect, though inconsiderable; but this effect varies according to the direction in which the force is applied; for if it were applied in an horizontal direction its weight would have less effect than if applied in raising a weight, because the

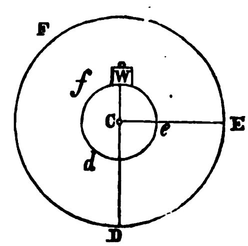
resistance of gravity would be less. To investigate these minute effects, varying with the angle which the line of direction makes with the horizon, would lead to a laborious inquiry, altogether incompatible with the scope of this compendium, and would hardly be worth the pains. But we may observe in general, that the power gained by the lever is the greater according as its direction approaches the line of vertical descent, because its force is aided by the action of gravity on the body moved in that direction.

CHAPTER II.

THE WHEEL AND AXLE.

The next instrument, or agent, which we are to consider, by which power may be applied to move bodies, is the wheel. A wheel is a cylinder, perforated at the centre by a concentric hollow circle, and moving freely round a fixed cylinder, called the axle, fitted into the central perforation.

Let the circle DEF in the fig. represent the outward rim of a wheel, of which the axle *edf* is at the centre C, CE will be a radius of the wheel, and C*e* will be a radius of the axle. Let this wheel be in a horizontal position to move freely on its axle without obstruction from friction.



Let a weight W be fixed to the wheel close to the axle, at a point in the circumference perpendicular to CE, and let any power applied at E move the rim of the wheel from E to D, then will the weight W be moved from W to *e*, and the velocity of the power is represented by the arc ED, and the velocity of the weight is represented by the arc W*e*; but these arcs are as the radii CE. C*e* (21, *New Supplement*); wherefore, the velocity of the power is to the velocity of the weight as CE to C*e*.

And by (chapter ix. page 91.), the moving force is $=$ the product of the weight into the velocity; that is, $F = WV$.

Let w represent the weight;

P the power applied at E;

L the radius of the wheel CE; and

l the radius of the axle Ce.

Theorem 1.

The weight is to the power as the radius of the wheel is to the radius of the axle; that is, $w : P :: L : l$. For the force of the weight $= wl$, and the force of the power $= PL$; and when these forces are equal and balance each other, $wl = PL$; and if these factors be made proportionals,

$$w : P :: L : l.$$

It is obvious by inspection, that the power of the wheel is that of a lever of the second kind, where the weight is placed between the fulcrum and the power; the centre of the axle being taken as the fulcrum; the rim of the wheel, the other end of the lever; and the rim of the axle, or the circumference of the inner circle of the wheel, the position of the weight. For we find in the case of the wheel (by theorem 1, above), that $w : P :: L : l$; and in the case of the lever of the second kind (theorem 2), $P : w :: l : L$; and conversely $w : P :: L : l$; hence the power of the wheel is analogous to that of the lever of the second kind, and we may (to save repetition) adopt the following formulæ, which we have demonstrated as to that kind of lever as belonging also to the wheel and axle.

$P = \frac{wl}{L}$; by theorem 3, chap. i. § 2.

$w = \frac{PL}{l}$; by theorem 4, ditto;

$L = \frac{wl}{P}$; by theorem 5, ditto;

$l = \frac{PL}{w}$; by theorem 6, ditto.

If any three of the four quantities, P, w, L, l, of the wheel are given, the remaining quantity may be found by the following table.

TABLE. CHAPTER II.

A Table of Formulæ for finding the power, weight, and radii of the wheel and axle constituting an equilibrium.

	GIVEN.	SOUGHT.	VALUES.
1	w, l, and L	P	$P = \frac{wl}{L}$.
2	P, L, and l	w	$w = \frac{PL}{l}$.
3	w, l, and P	L	$L = \frac{wl}{P}$.
4	P, L, and w	l	$l = \frac{PL}{w}$.

Scholium 1.

The axle of a wheel is sometimes considered as a smaller concentric wheel firmly fixed to the larger one, both moveable together round the common axis of both; but in this case, as well as in the detached wheel and axle, the weight is in effect fastened to the wheel at the circumference of the axle, and whether the axle is fixed to the wheel or not, this difference in circumstances makes no difference in the power acquired.

Scholium 2.

In carriages on wheels the *draught* is applied at the centre, but the power of the wheel acts at the circumference, viz. at the point of contact of the rim of the wheel with the

ground at the successive times of contact, and impels the weight at the circumference of the loose axle.

Problem 1.

A carrier by land, having goods to convey, weighing 15 tons, stows them into a waggon weighing 3 tons; the average radii of the wheels to the radii of their axles are as 18 to 1: how many horses must he employ to draw the waggon, supposing the draught of each horse to be equivalent to 4 cwt.?

Here w is given $= 15 + 3 = 18$ tons $= 18 \times 20 =$ 360 cwt., and l is given $= 1$, and L is given $= 18$, and P the power is sought.

By formula 1,

$$P = \frac{wl}{L} = \frac{360 \times 1}{18} = 20 \text{ cwt.};$$

and since each horse is capable of drawing 4 cwt. $\frac{20}{4} = 5$, is the number of horses required.

Problem 2.

If a weight of 25*lbs.* be suspended from the periphery of a wheel, whose radius is 10 feet, what weight will it support suspended from the circumference of its fixed axle, whose radius is 15 inches?

Here P is given $= 25$*lbs.*, L is given $= 10$ feet $= 120$ inches, and l is given $= 15$ inches, and w the weight is sought.

By formula 2,

$$w = \frac{PL}{l} = \frac{25 \times 120}{15} = 200 lbs.; \text{ the answer.}$$

Problem 3.

Upon a fixed axle, whose radius is 18 inches, it is re-

quired to support a weight of 5 cwt., by means of a weight of 56*lbs.*, suspended from the periphery of an upright wheel. What must be the radius of the wheel to accomplish this purpose exactly?

Here P is given $= 56$*lbs.*, $w = 560$*lbs*, and $l = 18$ inches, and L, the radius of the wheel, is sought.

By formula 3,

$$L = \frac{wl}{P} = \frac{560 \times 18}{56} = 180 \text{ inches} = 15 \text{ feet; the}$$

answer.

Problem 4.

An upright wheel, whose radius is 10 feet, supports with a power of 80*lbs.* at the circumference, a weight of 640*lbs.* suspended to the rim of its fixed axle: what is the radius of the axle?

Here L is given $= 120$ inches, and $P = 80$*lbs.*, and $w = 640$*lbs.*, and l is sought.

By formula 4, table,

$$l = \frac{PL}{w} = \frac{80 \times 120}{640} = 15 \text{ inches; the answer.}$$

CHAPTER III.

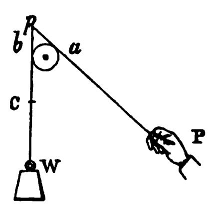
THE PULLEY.

Section 1.

The Fixed Pulley.

A fixed pulley is a small wheel with a grooved rim, which revolves round its axis, and is used as a contrivance or means whereby a string or cord, supporting a heavy body, may in effect slide or pass freely over a point near to which the pulley is placed.

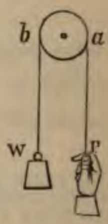
Let W be a weight suspended by a cord passing over the point *p*, and held in equilibrio by a power exerted at P.



If it were desired by the exertion of an additional power at P, to raise the weight from W towards the point *p*, it is found by experience, and is indeed very obvious, that if the string passed over the point *p* at the acute angle P*p*W, the power would be much obstructed, and the cord fridged and worn. To avoid this obstruction the pulley has been brought into practice, with a groove or volute along its rim to receive the cord as near the vertex of the angle as the pulley will fit

in; by which means the direction of the remaining parts of the cord aP. bW, is not altered.

When the pulley is so placed, it affords no advantage to the power P over the weight W; the power must be equal to the weight, in order to sustain it; that is, no power is gained by the pulley in this case. For if the weight was pulled up to the point c, the point a would descend an equal distance in the direction from a to P; wherefore, their velocities would be equal; and because the weight cannot be kept in equilibrium unless the forces are equal, $WV = Pv$; and because $V = v$, ergo, dividing by these equals $W = P$.



Hence, when the weight is kept in equilibrio by a power acting in the opposite direction, it is obvious that double the weight is thrown upon the pulley (or the pin or axis that supports it), than it would have to sustain, if the weight alone were suspended from the pulley or its axis.

But as the fixing of the pulley to sustain any ordinary weight is very easily accomplished by means of bolts and screws, this duplication of the weight upon the pulley is not a matter of consequence. However, it is obvious that this increase of the weight is not accompanied with any increase of the power P. In this, the most simple application of the pulley to practice, its use is only as a point over which the cord will slide easily, for the purpose of moving a body in a direction according to the exigency when the convenience of the operator requires that the requisite force should operate in a direction somewhat contrary or opposite to the direction in which he applies it. Thus, in the last figure, the power is applied in the downward direction from a to P; but it operates in the contrary direction, upward, from W to b.

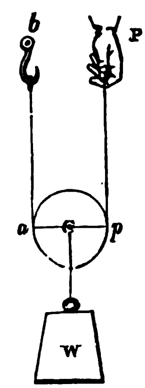
SECTION 2.

The Single Moveable Pulley.

Archimedes appears to have discovered, that the pulley possessed the property of increasing power upon the same principle as a lever of the second kind; and that in order to bring this property into action, it was necessary that the pulley should be moveable, and should be supported by a cord which would have the effect of a fulcrum.

It was found that the single moveable pulley had the effect of doubling the power, as follows:

Let W represent a weight suspended at the centre of a pulley C, whose diameter is *ap*, supported by a cord fastened to the hook *b*, passing in the groove of the pulley, and held up by the hand at P.



In this case the power is applied at *p*, one end of the diameter *ap*, which is the lever, and acts upon the weight at the centre *c*, and the other end of the diameter *a* is the fulcrum; so that the abscissa *a*C is the radius equal to half the diameter or lever *ap*.

Theorem 1.

When a weight suspended from a moveable pulley is in equilibrium the power is half the weight.

Let P represent the power; and
w the weight.

Then $P = \frac{w}{2}$.

For by the property of the lever, when the forces are in

equilibrio, $P : w :: \text{radius } aC : \text{diameter } ap :: \frac{1}{2} : 1$; (theor. 2, chap. i. § 2.) that is, (by doubling the last two terms) $:: 1 : 2$;

$$\text{that is, } P : w :: 1 : 2;$$

and because the products of the extremes and means are equal;

$$w = 2P;$$

and dividing these equals by 2, $\frac{w}{2} = P$.

Theorem 2.

When a weight suspended from a moveable pulley is in equilibrium, the weight is equal to double the power; that is, $w = 2P$.

This is demonstrated in theorem 1, wherein it is shown that $w = 2P$.

From these theorems we collect the two formulæ, $P = \frac{w}{2}$; and $w = 2P$; and therefore, if either P or w be given, the other may be found.

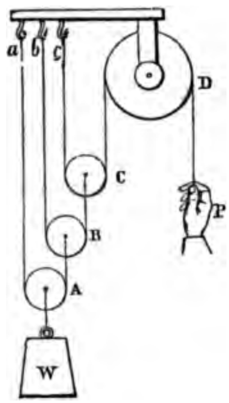
Section 3.

First Combination of Pulleys.

Since one moveable pulley has the effect of reducing any weight one half, another moveable pulley applied in reduction of the weight pressing on the first pulley would reduce the weight to one-fourth; a third pulley applied to the weight on the second, would reduce it to one-eighth; a fourth pulley to one-sixteenth; and so on. So that the power would be to the weight as 1 to 2^n; where n denotes such a power of 2, whose index is equal to the number of moveable pulleys used.

Any number of moveable pulleys may be made to act in aid of each other by the following method or contrivance.

Let a weight W be suspended from the centre of a moveable pulley A, by means of a cord fastened at one end to a hook *a*, and coming under and supporting the pulley A; the other end of the cord being fastened to the centre of another pulley B. Let the weight be considered as in equilibrio in this position; the cord from A to the centre of the pulley B will sustain half the weight W (by theorem 1, *Single Moveable Pulley*).



Also, considering the weight still in equilibrio, half the weight W being the pressure on the pulley B, the cord from B to the centre of pulley C will sustain half of this weight; that is, it will sustain half of half (or a quarter) of the weight W, $= \frac{1}{4} = \frac{1}{2^2}$; and in like manner the cord from the pulley C to the fixed pulley D (by which no power is gained), sustains $\frac{1}{2}$ of the weight borne by pulley C, that is, $\frac{1}{2}$ of $\frac{1}{4} = \frac{1}{2^3}$. If there were 4 moveable pulleys, a fourth pulley would reduce the ratio of the power to the weight as $\frac{1}{2^4}$ to 1; a fifth pulley to $\frac{1}{2^5}$; and so on: so that universally in this combination the power is to the weight as 1 to 2^n. Hence, in this case the power gained is a *geometrical* progression, in which 2 is the first term and the common ratio.

Theorem.

The power is equal to the weight divided by that power of 2, whose index is equal to the number of moveable pulleys; that is, $P = \frac{w}{2^n}$.

For $P : w :: 1 : 2^n$; and because the products of the extremes and means are equal,

$$P \times 2^n = w \times 1; \text{ that is, } P \times 2^n = w;$$

and dividing each of these equals by 2^n, $P = \frac{w}{2^n}$.

Hence, we derive the following Rule for finding the power requisite to sustain any given weight by this combination of pulleys.

Rule.—Divide the weight by that power of the number 2, whose index is equal to the number of pulleys; the quotient will be the power required.

Note.—The weight of the moveable pulleys is to be added to the weight to be raised or sustained.

Section 4.

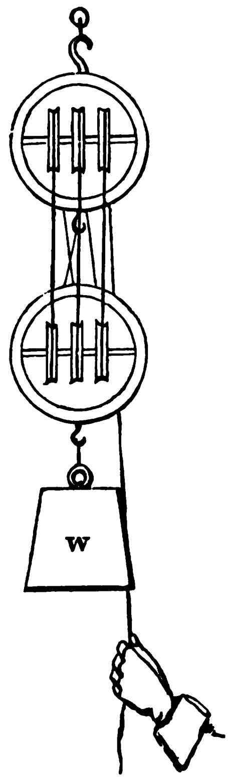
Of Block-Pulleys.

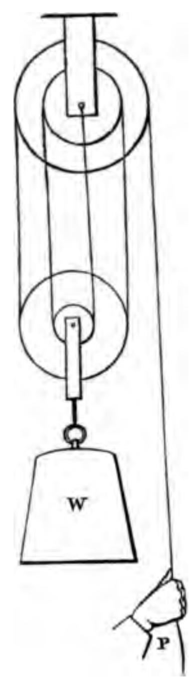
The combination of moveable pulleys is much more conveniently effected by means of block-pulleys, but with a less increase of force.

Block-pulleys are circular or oval blocks of wood, and are used in pairs. Each block is hollowed out so as to contain three or more pulleys, all fixed upon the same axis in the block.

The blocks contain an equal number of pulleys, and are placed vertically one above the other, so that the grooves of the pullies in one block are opposite to the grooves of the pulleys in the other block. The upper block is sus-

pended by a hook from a fixed bar or bolt. The weight to be raised is suspended by a hook fixed at the bottom of the lower block. A rope is fastened or attached by a hook to the bottom of the upper block, and passes into the groove of one of the lower pulleys, and thence into the





groove of the opposite pulley above, and thence alternately into the grooves of the lower and upper pulleys, until finally the rope passes over the last pulley in the upper block, and hangs down towards the weight. The power is applied at this end of the rope pulling vertically downward. The rope passing over the first upper pulley

acts upon the first lower pulley, and the power when applied raises the lower block by the joint effect of all the pulleys. The upper pulleys being all fixed do not increase the power; but the lower pulleys being moveable, act simultaneously as three distinct levers, by which the power is applied directly to the weight, each of which levers doubles the force of the power.

In this case one moveable pulley does not act as a lever upon another moveable pulley, (as in the first combination); but here the same power acts directly on all the moveable pulleys, and their effect is directly upon the weight. Hence, the effect of two pulleys in the lower block is double that of one pulley; the effect of three is treble; the effect of four would be quadruple, &c.

Hence, with the block-pulley, $P : w :: 1 : 2n$; where n represents the number of pulleys in the lower block. The power, therefore, which is gained by the block-pulleys is an *arithmetical* progression, in which 2 is the first term, and the common difference.

Theorem.

The power is equal to the weight divided by the product of 2 into the number of pulleys in the lower block; that is,

$$P = \frac{w}{2n}.$$

For we have seen that

$$P : w :: 1 : 2n;$$

and because the products of the extremes and means are equal,

$$P \times 2n = w \times 1 = w;$$

and dividing each of these equals by $2n$,

$$P = \frac{w}{2n}.$$

From this theorem we derive the following Rule for

finding the power requisite to sustain any given weight by block-pulleys.

Rule.—Divide the weight to be raised plus the weight of the lower block by twice the number of pulleys in the lower block; the quotient will be the power required.

EXAMPLES.

1. With a simple moveable pulley, what power will be sufficient to sustain 1 ton weight?

By theorem 1, *(Single Moveable Pulley),*

$$P = \frac{w}{2} = \frac{20\text{ cwt.}}{2} = 10\text{ cwt. or } \tfrac{1}{2}\text{ ton; the answer.}$$

2. With four moveable pulleys combined, as in section 3, what power will be sufficient to sustain the weight of 1 ton?

By the theorem *(First Combination of Pulleys,* § 3.)

$$P = \frac{w}{2} = \frac{20\text{ cwt.}}{2^4} = \frac{20 \times 112\mathit{lbs.}}{16} = \frac{2240}{16} = 140\mathit{lbs.}$$

3. With block pulleys, each block containing four pulleys, what power will be requisite to sustain 1 ton weight?

By the theorem (*Block Pulleys,* § 4.)

$$P = \frac{w}{2n} = \frac{2240\mathit{lbs.}}{2 \times 4} = 280\mathit{lbs.}\text{; the answer.}$$

4. With block pulleys, each block containing two pulleys, what power will be requisite to sustain 1 ton weight?

$$P = \frac{w}{2n} = \frac{2240}{2 \times 2} = 560\mathit{lbs.}\text{; the answer.}$$

5. With two moveable pulleys, combined as in section

8, what power will be required to sustain a weight of 1 ton?

$$P = \frac{w}{2^n} = \frac{2240}{2^2} = 560 lbs.; \text{ the answer:}$$

being the same power that would be requisite with two block-pulleys; because here $2^n = 2n = 2 \times 2$. In this case, therefore, block-pulleys, consisting of two each, are as powerful as two moveable pulleys used as in the first combination.

CHAPTER IV.

THE INCLINED PLANE.

(Art. 1.) An inclined plane is a hard, level, and immoveable plane surface at an angle with the horizon less than a right angle *below* the plane.

(2.) Thus the right line *ab* represents the side or *profile* of a plane inclined at the angle *bac*, below the plane (less than a right angle), with the horizon *ac*. The perpendicular *bc*, represents the altitude or height of the plane.

(3.) The *use* of the inclined plane is to raise or lift a heavy body upwards to the height of the plane. The *manner* in which it is used is by drawing or impelling the body up the plane from *a* to *b*. The *advantage* of it consists in its requiring a less force to impel the body from *a* to *b* on the plane, than to raise it perpendicularly upward to the altitude of the plane, without mechanical contrivance. Hence, power is gained by the inclined plane, but it is at the expence of space, and in the same ratio, as will appear hereafter.

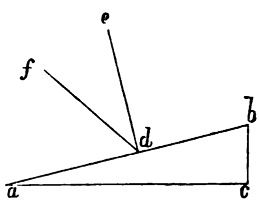
(4.) The inclined plane operates by means of its *resistance* to the action of gravitation. Because it is hard and immoveable, its force of resistance is considered greater than any force which can be opposed to it. Hence the action of the force of gravity, meeting with the superior resistance of the inclined plane, excites into operation a *reaction* from the plane equal (not to the *capacity* of its resistance, for that by the hypothesis is infinite, but) to the opposing action of the gravitating body. For we have seen (chap. i. part i.) that when the resistance is greater than the opposing force, the reaction of the resistance is *equal* to the action of the force, not greater than the action.

(5.) The inclined plane and gravity may be considered as two forces acting simultaneously on the body. If these forces acted in directly opposite directions, the *action* of gravity would absorb, that is, would be equal to the whole of its *force;* which we have seen (art. 24, chap. x. part i.) it would not be if the direction of the forces were at any angle with each other. And because the plane would oppose to the gravitating body a reaction equal to its action, the reaction may be considered as an equivalent active force emanating from the plane in the same direction with its reaction, and equal to the force of gravity; and because the body is simultaneously acted upon by these two equal and opposite forces it would remain at rest (art. 2, part. i. chap. vi.); but a round ball, or a body on wheels, is invariably found to roll down an inclined plane, however small may be the angle of inclination, contrary to the hypothesis; wherefore the direction of the reaction of any inclined plane cannot be directly opposite to the action of gravitation, which is at right angles with the horizon.

(6.) But if the upper end of the plane were sunk so as to bring it to a level with the horizon, then the ball or body

on wheels would remain at rest; and, therefore, these equal forces would, in that horizontal position of the plane, act in directly opposite directions, and the action of the plane would be at right angles to the horizon. In this case, therefore, the direction of the reaction of the plane is upward, at right angles to the plane itself; and it will be found to be also at right angles with the plane when it is inclined to the horizon, at any angle whatever.

(7.) For let *ab*, be the surface of a plane inclined to the horizon *ac*, at the angle *bac*, and let *de* be drawn at right angles to the plane *ab*; and if from the point *d*, the reaction of the plane is not in the direction from *d* to *e*, at right angles to *ab*, let it be in any other direction *df*, and let *df* represent a steel rod.



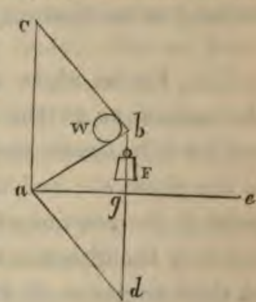
At the point *f*, let a force be applied to the rod *fd*, impelling the other end *d* against the surface of the plane in the direction *fd*; and because the reaction of the plane is, from its hardness and immoveability, equal to the action of the force applied, however great it may be, and is (by the hypothesis) in a directly opposite direction, the end of the rod would remain immoveable at *d*: but this is contrary to invariable experience; for under these circumstances the end of the rod would move from *d* to *b*, which falsifies the hypothesis; and, consequently, the direction of the reaction of the plane is not in the direction *df*, which is at any other angle with the plane than a right angle; that is, *the direction of the reaction of an inclined plane is outward at right angles to the plane.*

Theorem 1.

The power required to sustain a body in equilibrio on

an inclined plane is equal to the resultant of the force of gravity, and of the force of the inclined plane.

Let *ab* be an inclined plane upon which the body W is placed at the upper end *b*, and let F, the sustaining power, be a weight connected with the body W, and suspended by a cord passing over a fixed pulley at *b*, the upper end of the inclined plane. Let the line *bd*, perpendicular to the horizon *ae*, represent the force and direction of gravity, or of the weight of the body W at the point *b*, and let *b*C, perpendicular to the inclined plane, represent the direction of the force, or reaction of the plane at the point *b*. If the weight F were detached from the body W, the body would roll or slide down the plane *ba*, from the joint effect of the two forces, gravity and the plane, acting simultaneously upon it; and consequently, the surface of the plane *ab* is the resultant of those two forces, that is, *ab* is the diagonal of a parallelogram of which two contiguous sides represent the two simultaneous forces, (art. 7, part. ii. chap. x.) the direction of which forces is given *in position*. Let *bd* represent *the space* that would be described by a body falling freely from *b*, by the force of gravity in 1″; then *bd* represents the force of gravity at the point *b*, both in position and *magnitude;* and the surface of the plane *bd* represents, both in position and magnitude, the joint effect of the force of gravity, and of the inclined plane, the other force.



Draw *a*C equal and parallel to *bd*, and complete the parallelogram *bda*C; then will the right line *b*C represent the force of the inclined plane both in position and magnitude.

Because *ab* and *bd* are given in position and magnitude, the right line *ad*, which joins their extremities *a* and *d*,

is also given in position and magnitude; and because (1 *Euclid*, 34.) *b*C is equal to *ad*, *b*C is also given in position and magnitude.

Hence, if the forces were applied successively, the body would move from *b* to *d* by gravity in 1″, and from *d* to *a* in the next second; but because these two forces act simultaneously, the body W would move from *b* to *a* in 1″, describing the diagonal *ba* of the parallelogram *bda*C; *ba* is, therefore, the resultant of the two forces. Wherefore, the power required to sustain the body at *b*, must be such as if applied at the point *a*, would cause the body to move from *a* to *b* in 1″. Let the weight F be such as would draw the body from *a* to *b* in 1″, then there would be three forces represented by the sides of the triangle *abd* acting upon the body simultaneously at the point *b*. Wherefore, (by art. 19, chap. x. part. i.) the body would be sustained in equilibrio at the point *b*; but *ab*, which represents the power of the weight F is the resultant of the force of gravity, and of the force of the inclined plane.

Wherefore the power required, &c. Q. E. D.

Coroll. 1. The force of gravity on an inclined plane is represented by the space which would be described by gravity in the time during which the body moves down the plane.

Coroll. 2. The force of an inclined plane is to the force of gravity as the base to the length of the plane.

Theorem 2.

The power required to keep any weight in equilibrio on an inclined plane is to the weight as the altitude or height of the plane is to its length.

Because of the parallels *bd* and C*a* (*fig. above*) the alternate angles *abd* and *ba*C are equal; and the angle *agb* is a right angle by the construction, and the angle *ab*C is a right angle (by art. 7.); wherefore, two of the angles

in the triangle agb are equal to two of the angles in the triangle abC; hence, the remaining angle gab, is equal to the remaining angle aCb; and, consequently, the two triangles agb and abC are similar, and their homologous sides are proportionals.

But the hypothenuse aC represents the weight, and the shortest side ab of the triangle abC represents the power; hence, $P : w :: ab : aC$; and because the triangles are similar; $ab : aC :: gb : ab$; and by equality of ratios, $P : w :: gb : ab$; but gb represents the altitude of the plane, and ab represents its length; wherefore the power required, &c. Q. E. D.

Let P represent the power applied;
w the weight;
h the height or altitude of the plane (gb); and
l its length, (ab).

Theorem 3.

The power is equal to the product of the weight into the height divided by the length of the inclined plane; that is,

$$P = \frac{wh}{l}.$$

By theorem 2, $P : w :: gb : ab$; that is, $P : w :: h : l$; and because the products of the means and extremes are equal,

$$Pl = wh; \text{ and dividing by } l,$$

$$P = \frac{wh}{l}.$$

Coroll. $Pl = wh$.

Theorem 4.

The weight is equal to the product of the power into the length divided by the height; that is,

$$w = \frac{Pl}{h}.$$

By coroll. theorem 3, $Pl = wh$; and dividing by h,

$$\frac{Pl}{h} = w.$$

Theorem 5.

The length of the inclined plane is equal to the product of the weight into the height divided by the power; that is,

$$l = \frac{wh}{P}.$$

By coroll. theorem 3,

$Pl = wh$; and dividing by P,

$$l = \frac{wh}{P}.$$

Theorem 6.

The height of the inclined plane is equal to the product of the power into the length divided by the weight; that is,

$$h = \frac{Pl}{w}.$$

By coroll. theorem 3,

$Pl = wh$; and dividing by w,

$$\frac{Pl}{w} = h.$$

From the last four theorems we collect the following formulæ:

$$P = \frac{wh}{l}; \qquad l = \frac{wh}{P};$$

$$w = \frac{Pl}{h}; \qquad h = \frac{Pl}{w}.$$

From these formulæ, if any three of the four quantities, P, w, l, and h, be given, the fourth may be found, as is shown in the following table:

TABLE. CHAPTER IV.

A Table of Formulæ for finding the lengths and heights of inclined planes, and the weights and powers constituting an equilibrium.

	GIVEN.	SOUGHT.	VALUES.
1	P, h, and l	P	$w = \frac{wh}{l}$.
2	P, l, and h	w	$w = \frac{Pl}{h}$.
3	w, h, and P	l	$l = \frac{wh}{P}$.
4	P, l, and w	h	$h = \frac{Pl}{w}$.

EXAMPLES.

1. Required the power necessary to sustain a weight of 1 ton on an inclined plane, whose length is 10 yards, and height 3 yards.

Here w is given $= 20$ cwt., $h = 3$ yards, and $l = 10$ yards, and P, the power, is sought.

By formula 1,

$$P = \frac{wh}{l} = \frac{20 \times 3}{10} = 6 \text{ cwt.; the power required.}$$

2. What weight will a power of 50*lbs.* sustain on an inclined plane, whose length is 12 yards, and height 1 foot?

Here P is given $= 50$*lbs.*, $l = 12$ yards $= 36$ feet, and $h = 1$ foot, and w is sought.

By formula 2,

$$w = \frac{Pl}{h} = \frac{50 \times 36}{1} = 1800 lbs.\text{; the answer.}$$

3. With a power of 40*lbs.*, what should be the length of an inclined plane ascending 2 feet in perpendicular height, to sustain a weight of 360 *lbs.* upon the plane?

Here w is given $= 360$*lbs.*, $P = 40$, and $h = 2$ feet, and l, the length of the inclined plane, is sought.

By formula 3,

$$l = \frac{wh}{P} = \frac{360 \times 2}{40} = 18 \text{ feet; the answer.}$$

4. What should be the greatest inclination of a plane 12 yards long, to allow of its sustaining a weight of 560*lbs.*, with a power of 40*lbs.*?

Here w is given $= 560$*lbs.*, $l = 36$ feet, and $P = 40$*lbs.*, and h is sought.

By formula 4,

$$h = \frac{Pl}{w} = \frac{40 \times 36}{560} = 2\tfrac{4}{7} \text{ feet; the answer.}$$

CHAPTER V.

THE WEDGE.

(Art. 1.) A wedge is a triangular prism of wood or iron, and is used in riving asunder materials, by introducing its edge into a cleft, and then by striking it on the head with a hammer or mallet by which it tends to widen the cleft at each blow. The *direction* of the force which drives the wedge is perpendicular to the head, and the planes of the two sides are inclined to the direction of the driving force. Hence the wedge is a combination of two *moveable* inclined planes, which operate (not by their passive resistance, as does the *immoveable* inclined plane, but) as an impulse, or active force on the bodies or parts to be moved. The *resistance* to be overcome is the cohesion of the parts before severance.

(2.) If, instead of the wedge being driven against the body, the body were impelled against the wedge, the effect of the *action*, or rather, in that case, the *reaction*, of the wedge upon the parts of the body would be the same. In each case the loci or paths of the action of the edge would be *along the sides* of the wedge. If the wedge were driven, its motion would be perpendicular to the head, and the parts separated would be lateral, or sideways; if

the body were driven against the wedge (now considered as fixed) the parts of the body only would move along the planes or sides of the wedge.

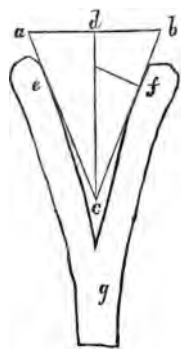
(3.) Hence, each side of the wedge possesses the properties of an inclined plane of the same altitude, and (by art. 7, chap. iv.) the *direction* of the action of the wedge is at right angles to its sides. But we are to observe, that the word *equilibrium*, when applied to the wedge, does not mean that state of rest in which the wedge remains from its hardness during the intervals of the blows of the hammer, but that state of rest which the wedge would maintain if it were struck with a force barely equal to the resistance, and not sufficient to drive it forward. We are also to observe that the relation of *weight* to the inclined plane is the same as that of *resistance* to the wedge.

CASE I.

When the body is moveable on both sides.

Theorem 1.

The power of a wedge in separating the parts of a body is to their resistance as half the width of the head of the wedge is to one of its sides.



Let *abc* be an *isosceles* wedge used in separating the parts *e* and *f*, or in riving asunder the trunk *g*; then will the power of the wedge be to the resistance as *db*, half the head, is to the side *bc*.

Draw *cd* perpendicular to *ab*, *db* will be the altitude of the plane or side *cb*, and will be equal to *da* (1 *Euclid*, 26.); that is, *db* will be

half the width of the head of the wedge, and *bc* is one of the sides of the wedge. But (by theor. 2, chap. iv.) the power required to keep a weight in equilibrio on an inclined plane, is to the weight as the altitude of the plane is to its length; wherefore, (by art. 3 in this chapter) the power of the side *bc* is to the resistance as *bd* to *bc*.

In like manner, the power of *ac*, the other side of the wedge, may be demonstrated to be as *ad* to *ac*.

But the power of both sides of the wedge (that is, double the power), is to double the resistance, (that is, to the resistance against the whole wedge), as half the power, *bd* or *da*, to half the resistance *bc* or *ac*; wherefore, the power of the wedge in separating the parts of the body is to their resistance, &c. Q. E. D.

Let P represent the power;
R the resistance;
H the length of the head of the wedge; and
S one of its inclined sides.

Theorem 2.

The power of the wedge is equal to the product of the resistance into half the head divided by one of the inclined sides; that is,

$$P = \frac{R\frac{H}{2}}{S}.$$

By theorem 1, $P : R :: \frac{H}{2} : S$; and because the product of the means is equal to the product of the extremes,

$$PS = R\frac{H}{2};$$

and dividing each of these equals by S,

$$P = \frac{R\frac{H}{2}}{S}.$$

From theorem 2 we derive the following rule for finding the power necessary to be applied to a wedge to make it equivalent to any given weight or resistance.

Rule.—Multiply the given resistance or weight into the length of half the head of the wedge, and divide the product by the length of one of the inclined sides; the quotient will be the power required.

EXAMPLE.

The length of one of the inclined sides of a wedge being 12 inches, and its head 4 inches, what power would be required to separate two moveable bodies whose resistance together is 240*lbs.*?

Here R is given $= 240lbs.$, $H = 4$, and $S = 12$, and P, the power, is sought.

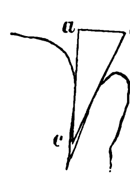
By theor. 2,

$$P = \frac{R\frac{H}{2}}{S} = \frac{240 \times 2}{12} = \frac{480}{12} = 40lbs.; \text{ the answer.}$$

CASE II.

When only one of the bodies is moveable.

In this case the whole of the wedge is applied to move the moveable body. It is, in effect, therefore, a half wedge, or a wedge, one of whose sides is at right angles with its head. Consequently, in this case,



$$P = \frac{RH}{S}.$$

EXAMPLE.

With a wedge whose head is equal to 3 inches, and the length of its inclined side is 12 inches, what force would be sufficient, by driving the wedge, to rive asunder a por-

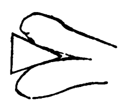
tion of a precipitous rock, which required a power of 4 tons to sever it from the main rock?

Here R is given = 4 tons = 80 cwt., H = 3, and S = 12, and P is sought.

By the formula,

$$P = \frac{RH}{S} = \frac{80 \times 3}{12} = 20 \text{ cwt.} = 1 \text{ ton; the ans.}$$

In this case power would be lost, if the wedge were struck in a direction inclined to the immoveable side.

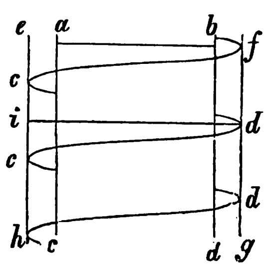
Note.—The power of a wedge is always the greatest when used in a vertical position, thus  and least, when used in a horizontal position, thus,  because the action of gravity opposes the action of the wedge less in the vertical than in the horizontal position.

CHAPTER VI.

THE SCREW.

(Art. 1.) A screw consists of a solid cylinder fitted into a hollow cylinder of equal or a little greater diameter; upon which solid cylinder a spiral projecting worm or thread is wound round and fixed; and in the hollow cylinder a groove is made in the same spiral direction, so that when the solid cylinder is turned round on its axis, the worm or thread (which constitutes the screw) proceeds into the groove or flute of the hollow cylinder, whether the solid cylinder is turned in one direction or the other, backward or forward, provided the perpendicular distances between the threads are equal to each other.

Fig. 1.



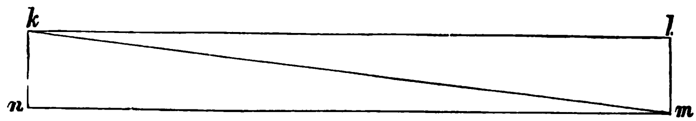
(2.) Let *abcd* (fig. 1.) represent the internal solid cylinder, on which the worm or thread is wrapped and fixed, and the lines *cf* and *gh* represent an imaginary cylinder, within which the outer edges *cc*, *dd*, of the screw or worm are barely in contact. Let *di* be a diameter of a circle described by the horizontal section of this imaginary cylin-

der, and let *fd* represent the perpendicular distance between the spirals.

(3.) Let the right line *kl* (fig. 2.) be equal to the circumference of the imaginary cylinder, whose diameter is *di*; and let the right line *lm*, drawn perpendicular to *kl*, be equal to *df*, the distance between the spirals; the hypothenuse *km* will be equal to the length of the spiral between *d* and *f*.

Complete the parallelogram *k l m n* (fig. 2.), and apply it

FIG. 2.



to the surface of the imaginary cylinder *efgh*, so that *l*, one point of the parallelogram *klmn*, may coincide with the point *f*, and that the side *lm* may fall upon the right line *fd*, and because *lm* is $=$ *fd*, the point *m* will coincide with the point *d*. Let the parallelogram *klmn* be supposed to be wrapped horizontally round the imaginary cylinder, *efgh*, then, because the side *mn* is equal to the circumference of the imaginary cylinder, *n* will coincide with *d*, and *k* will coincide with *f*, because *kn* $=$ *lm* $=$ *fd*; that is, *m*, one end of the hypothenuse *mk*, will coincide with *d*, and *k*, the other end of the hypothenuse *mk*, will coincide with *f*; and they would coincide with the spirals in any other similar position; wherefore, the hypothenuse *mk* is equal to the length of the external rim of the spiral in one revolution.

But the hypothenuse *km* is inclined to the horizon *mn*, at the angle *kmn*, which is, therefore, the angle of inclination of the spiral to the horizon; and the surface of the spiral which sustains the weight, is, therefore, an inclined plane, and possesses the mechanical properties of that instrument or power. (*Whewell's Mechanics*, p. 51.)

(4.) Hence, because the power of the inclined plane is to the weight which it will sustain in equilibrio, as its height is to its length (by theorem 2, chapter iv.); therefore, the power of the screw is to the weight which it will keep in equilibrio, as the distance between two contiguous threads is to the length of the spiral or extreme edge of the screw, between the points from which the distance between the two contiguous threads was taken; that is, as the perpendicular $lm = df$ to the hypothenuse $km \doteq$ the spiral dcf.

(5.) The length of the whole spiral or screw is to the perpendicular height of the cylinder as the spiral between the points f and d ($= km$) is to the distance df between the two contiguous threads; for the length of the whole screw consists of all the equal distances, fd, dd, &c.; and whatever be the number of distances, there will be as many spirals between them, each equal to the first spiral; and because magnitudes are as their equimultiples (5 *Euclid*, 15.), the height of the screw is to the length of its spiral as the perpendicular distance between any two contiguous threads is to the length of the spiral between the points in those threads from which the distance was taken *.

(6.) Let s represent the length of the spiral between two contiguous threads;

h the distance between two contiguous threads;

c the circumference of the imaginary cylinder of the screw;

w the weight sustained by the screw; and

P the power applied to maintain the equilibrium.

* The distance between two contiguous threads, and the length of the spiral between them may be measured with more ease and accuracy than the height of the whole screw, and the length of the whole spiral; which is the reason why the former ratio is preferred and adopted in these investigations.

Hence, by art. 4, P : w : : h : s; and, because the products of the means and extremes are equal, $Ps = wh$.

Theorem 1.

The length of the spiral of any screw between two contiguous threads is equal to the square root of the sum of the squares of the circumference of the screw, and of the distance between two contiguous threads; that is,

$$s = (h^2 + c^2)^{\frac{1}{2}}.$$

For, as we have seen the perpendicular lm (fig. 2.) represents h, the distance; the hypothenuse km represents s the spiral, and the base mn represents c the circumference of the screw; but the triangle klm is a right-angled triangle; and (by 1 *Euclid*, 47.) the square of the hypothenuse is equal to the sum of the squares of the base and perpendicular; that is $s^2 = h^2 + c^2$; and extracting the square root of these equals,

$$s = (h^2 + c^2)^{\frac{1}{2}}.$$

From this theorem we obtain the rule for solving the next problem.

Problem 1.

To find the length of the spiral, the circumference of the imaginary cylinder and the distance between two contiguous threads of the screw being given.

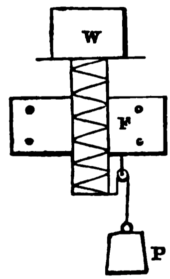
Rule.—Square the circumference and the distance; add the squares together, and extract the square root of their sum; that square root will be the length of the spiral as required.

[This rule is obviously only an enunciation of the operations denoted by the equation $s = (h^2 + c^2)^{\frac{1}{2}}$.]

Problem 2.

Let a weight W press upon an internal screw, as in fig. 3. Let this screw be inserted in an external fixed screw F to fit; and let the interlacing ribs be as in the figure. To counteract the pressure of the weight W downward, let another weight P be suspended at a pulley fixed in the external screw F, so as to tend to raise the internal screw vertically upward in the direction opposite to the weight. This suspended weight may be considered as the power applied to the screw to sustain the weight in equilibrio. Let it be required to find P the power.

FIG. 3.



By art. 6, $Ps = wh$; and dividing by s,

$$P = \frac{wh}{s}.$$

EXAMPLE.

Upon a screw, whose spiral between any two contiguous threads is 36 inches, and the distance between the two contiguous threads is 4 inches, what power will be required to keep in equilibrio a weight of 360*lbs.* pressing on the screw?

Here s is given $= 36$ inches, $h = 4$ inches, and $w =$ 360*lbs.*; and P, the power, is sought;

Because $Ps = wh$; dividing by s,

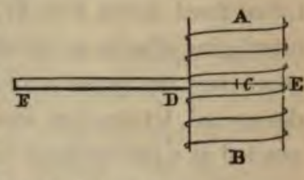
$$P = \frac{wh}{s} = \frac{360 \times 4}{36} = 40 lbs.$$

The screw is used principally for the purpose of communicating pressure upon bodies as in a press. To effect this purpose a horizontal bar is sometimes fixed to the internal cylinder so as to turn the internal screw round.

This addition of the bar to the screw renders it a

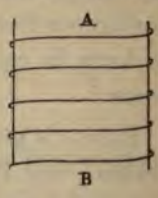
machine, in which the power gained by the action of the screw is increased by the action of the lever; that is, it is a combination of the screw and lever.

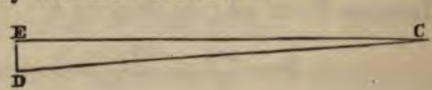
Let AB be a perpendicular screw, and FD, the arm of the bar, at right angles to the perpendicular sides of the screw. Let DE be the diameter of the screw, and DC the radius; the diameter and radius being both horizontal.



In this combination of the two powers of the screw and lever, acting in aid of each other, we may consider, in the first place, the separate effect, of each.

1. And first, with regard to the screw. The power being applied horizontally is not in the direction of the spiral, which being an inclined plane has an angle of inclination to the horizon, that is, to the circumference of the screw. Let EC (see *fig. below*) be equal to the circumference of the screw, and DC be equal to the spiral between two contiguous threads, and ED be equal to the perpendicular distance between any two contiguous threads; then the action of the power, which, if applied in the direction of the spiral DC, would be equal to the weight on the spiral, and would sustain it, is as EC to DC; and, therefore, the power gained by the screw is not as h to s, but as h to c; that is, not as the distance between the threads to the length of the spiral between them, but as the distance between the threads to the diameter of the external cylinder contiguous to the rim of the spiral; (art. 24, chap. x. part i.) for EC is the base, and DC is the hypothenuse, of the right-angled triangle CED.





Hence, in consequence of this horizontal application of the power,

$$P : w :: h : c; \text{ and } Pc = wh.$$

From which equation, if any three of the four quantities P. *w*, *h*, and *c*, are given, the fourth quantity may be found, viz.

$$P = \frac{wh}{c}; \; c = \frac{wh}{P}; \; w = \frac{Pc}{h}; \; h = \frac{Pc}{w}.$$

2. Having, by the first of these formulæ, found the value of $P = \frac{wh}{c}$ (which value is to be understood of the power, as applied at the diameter of the external cylinder), we are now to consider the further acquisition of power gained by the lever, of which the bar FD is the arm. (*Fig.* 1. p. 164.)

The lever in this case is a straight lever of the second kind, in which the weight is between the fulcrum and the power. The power applied to the screw is now become the weight to which this lever is applied. The power is applied to the lever FC at F; and C, the centre of the internal screw, is the fulcrum; and the weight upon the lever (which is the power of the screw) is at D, and CD is the abscissa between the fulcrum and the weight.

By the notation of the lever of the second kind, P represents the power applied at F; L the whole lever FC, *w* the weight, and *l* the abscissa CD, between the fulcrum and the weight.

And (by theorem 3, § 2. chap. i. part ii.)

$$P = \frac{wl}{L};$$

that is, the power is equal to the product of the weight into the abscissa CD, divided by the length of the whole lever CF.

By means of the formulæ for the screw and lever, we might solve the following

Problem.

To find the power of a screw worked by a lever of 24 inches, reckoning from the middle point of the inner cylinder, the distance between the threads being 2 inches, the length of the horizontal circumference 20 inches, and the weight to be held in equilibrium 40*lbs.*

Here, as to the screw,

$$\begin{aligned} h \text{ is given } &= \ \ 2 \\ c \ \ \ldots\ldots \ &= 20 \\ \text{and } w \ \ \ldots\ldots \ &= 40 \end{aligned}$$

and P is sought.

By the formula, as to the screw (p. 165.),

$$P = \frac{wh}{c} = \frac{40 \times 2}{20} = 4 lbs.$$

And as to the lever; here

$$\begin{aligned} w \text{ is given } &= \ \ 4 \\ L \ \ \ldots\ldots \ &= 24 \\ \text{and } l = \frac{20 \div 3{\cdot}1415}{2} &= 3{\cdot}183.* \end{aligned}$$

By the formula as to the lever,

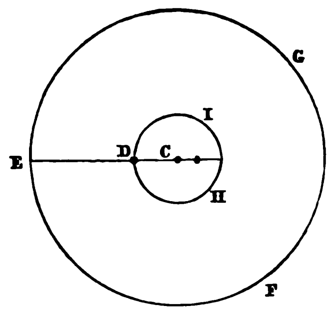
$$P = \frac{wl}{L} = \frac{4 \times 3{\cdot}183}{24} = {\cdot}5305 lb.;$$

that is, the screw and lever combined will, with a power of ·5305*lb.* (or rather more than half a pound) hold in equilibrio a weight of 40*lbs.*; to sustain which weight the screw

* The circumference is to the diameter as 3·1415 to 1, very nearly; consequently, if the circumference be 20, the diameter is $= \frac{20}{3{\cdot}1415}$; and l is the radius, that is, half this diameter.

alone would have required a power equivalent to $4lbs.$, if applied at the rim of the screw.

These operations may be much simplified by considering the length of the lever EC and the abscissa DC as the radii of two concentric circles EFG and DHI, the circumferences of which are as the radii EC. DC (prop. 19 and 20, *New Supplement to Euclid*); and if we substitute the circumferences for the radii, L will represent the circumference EFG, and l the circumference DHI; but c (in the Notation) also represents the circumference DHI; wherefore c may be substituted for l.



By theorem 2, chap. i. § 2, part ii.

$$P : w :: l : L;$$

and substituting c for l,

$$P : w :: c : L;$$

in which expression P represents the power of the lever; and because the power acting on the screw is the weight acted upon by the lever, if p represents the power of the screw; then $P : p :: c : L$; but by the formula as to the screw,

$$p : w :: h : c;$$

wherefore, if these six quantities be taken two and two in a cross order, that is, as P to p so is c to L, and as p to w so is h to c; then, *ex æquo perturbato*, $P : w :: h : L$; that is, the power of the lever and screw combined is to the weight pressing on the screw as the distance between the threads is to the circumference EFG, which (as we have seen above) is represented by L.

P	p	w
h	c	L

Because the circumference is to the diameter of any

circle as 1 to 3·1415, &c. (which is their constant ratio), the lever EC being half the diameter of the circle EFG; therefore, 2 EC $\times$ 3·1415 $=$ the circumference of the circle EFG. Hence if EC, the length of the lever be given, the circumference EFG may be found.

Considering, therefore, L as representing the circumference EFG in this case of combination of the screw and lever as a machine,

Because, $P : w :: h : L$; $PL = wh$.

Hence, we obtain the following formulæ,

$$P = \frac{wh}{L};\ L = \frac{wh}{P};\ w = \frac{PL}{h};\ \text{and}\ h = \frac{PL}{w}.$$

From which, if any three of the four quantities, P, L, w, and h, be given, the fourth may be found.

Thus, in the above example, or problem,

$$\begin{aligned} L \text{ is given } &= 48 \times 3{\cdot}1415 = 152{\cdot}792. \\ w \ \ldots\ldots\ &= 40 \\ h \ \ldots\ldots\ &= 2 \end{aligned}$$

and P is sought.

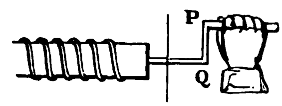
By the formula,

$$P = \frac{wh}{L} = \frac{40 \times 2}{152{\cdot}792} = {\cdot}5235 lb.$$

The difference between this result and the former, which is very inconsiderable, being but $\frac{7}{1000}$ of 1*lb.*, is occasioned by our not extending the constant quantity 3·1415, &c. to more places of decimals, by the omission of which the result is diminished in this solution much more than in the other.

Hence it appears, that when a lever is used to turn the screw, the length of the diameter of the cylinder, in which the rim of the spiral moves, is immaterial, whether it be longer or shorter; for the lever has the effect of making its own diameter become the diameter of the screw; and, consequently, the quantity *l* becomes exterminated out of the equation.

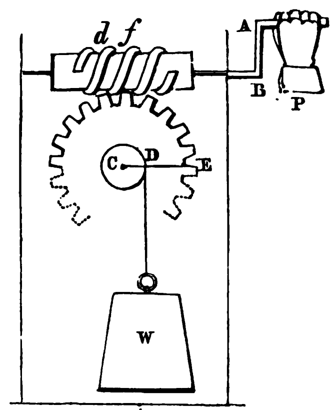
If a *winch* is used to turn the screw, the arm PQ between the shoulder Q, and elbow P of the winch becomes the radius of the cylinder of the screw, and is, therefore, half its



diameter, which diameter, being multiplied by the constant quantity 3·1415, &c., becomes in effect the horizontal circumference of the outer cylinder in which the rim of the spiral moves. The winch acts in aid of the screw as a lever, whose length is PQ, and whose direction is parallel to the circumference of the screw.

As the winch is used in a horizontal position, the screw which is turned by the revolution of the winch in the same horizontal position is not adapted in that position to be applied as the proximate or immediate means to raise a weight; the direction of the action of the weight being at a right angle to that of the action of the screw; but the action of the screw, when turned by a winch, may be applied to raise a weight by the intervention of a toothed wheel, connecting, by means of its axle, the screw with the weight; which purpose being accomplished, the winch, screw, and toothed wheel, form a *machine*, in which the lever, screw, and the wheel and axle, act in combination.

Let *df* be a screw, called commonly an endless screw, though it would be more intelligibly termed a continually acting screw, turned by the winch AB, and connected with the toothed wheel E, and its axle D with the weight W; the winch, screw, and toothed wheel, being all firmly fixed in their positions, but so as to leave them to perform their rotatory motions freely.



We may find the power of this machine in detail, by first finding the *power* of the wheel and axle, and then finding the *power* of the screw and winch combined, treating the *power*, of the wheel and axle, as the *weight* upon the screw. Thus, by the table, chap. ii., P (the power of the wheel and axle) $= \frac{wl}{\text{L}}$; and if this power, so found, be represented by p, and if p be substituted for w, the weight, in the formula for the screw and lever, P (the power applied at the handle of the winch) is $= \frac{ph}{\text{L}}$ (p. 168.)

First. Let us find the power of the wheel and axle.

Let EC, that is, $\text{L} = 18$ inches.

DC, that is, $l = 4$ inches, and

$w = 4{\cdot}54$ tons $= 10178{\cdot}46$*lbs.*

Then, by the formula of the wheel and axle, (table, chap. ii.) $\text{P} = \frac{wl}{\text{L}}$; that is,

$$\text{P} = \frac{10178{\cdot}46 \times 4}{18} = 2261{\cdot}88lbs.; \ = p \text{ being the}$$

power of the wheel and axle, to be now considered as the weight upon the screw.

Next we are to find the power to be applied to the winch to sustain in equilibrio $p = 2261{\cdot}88$*lbs.*

Let AB $= 12$ inches, L will be $= 24 \times 3{\cdot}1415 = 75{\cdot}396$ inches.

$w = p = 2261{\cdot}88$*lbs.*

$h = 1$ inch.

Then, by the formula of the screw and lever combined, p. 168,

$$\text{P} = \frac{wh}{\text{L}} = \frac{2261{\cdot}88 \times 1}{75{\cdot}396} = 30lbs.; \text{ the power to}$$

be applied to the winch to sustain a weight of 4·54 tons suspended to the axle at D.

But we may simplify this process by altering the notation, viz. by taking p (instead of P) as representing the

power of the wheel and axle, and C (instead of L) as representing the circumference of the circle described by the handle of the winch, and P, as before, as representing the power of the screw, and lever, or winch, combined; and L as representing the length of the radius of the wheel and axle (taken together).

Thus, as to the screw and winch together $P : p :: h : C$; $\therefore PC = ph$;
And as to the wheel and axle .. .. $p : w :: l : L$; $\therefore pL = wl$.

It being our object to avoid the intermediate operation of finding the weight upon the screw represented by p, the power upon the wheel; this object may be accomplished by finding the two equivalent values of p, in the two equations $PC = ph$, and $pL = wl$.

Thus, in the first equation, dividing by h, $p = \frac{PC}{h}$;

And, in the second equation, dividing by L, $p = \frac{wl}{L}$;

Wherefore, $\frac{PC}{h} = \frac{wl}{L}$; and clearing the fractions,

$$PCL = wlh; \text{ and } w = \frac{PCL}{lh}.$$

From which expression p is exterminated; and we can thereby find by one operation the weight held in equilibrium by the power (30*lb.*) acting at the handle of the winch,

$$w = \frac{PCL}{hl};\ \frac{30 \times 75{\cdot}396 \times 18}{1 \times 4} = 10178{\cdot}46 lbs.$$

$$\text{Also } P = \frac{wlh}{CL} = \frac{10178{\cdot}46 \times 4 \times 1}{75{\cdot}396 \times 18} = 30 lbs.$$

CHAPTER VII.

THE ASCENT AND DESCENT OF BODIES OVER A FIXED PULLEY.

(Art. 1.) If two weights, one of 2*lbs.*, and the other 1*lb.*, are suspended over a fixed pulley, the 2*lbs.* weight will preponderate and descend, drawing the other weight upwards. If the 2*lbs.* weight descended freely without any obstruction, it would describe by the force of gravity 16 feet in 1″; and because momentum varies as the product of the weight into the velocity; or M $\propto$ WV (art. v. theorem 1.) the momentum of the 1*lb.* weight may be taken *comparatively* as $16 \times 1 = 16$. Let the 1*lb.* weight be doubled; then if its momentum continues the same as before, that is, 16, its velocity would be diminished in the inverse ratio $= \frac{16}{2} = 8$, because $8 \times 2 = 16$. And because the resistance of bodies moving in opposite directions is as their momenta, the resistance of the 1*lb.* weight to the 2*lbs.* weight is equal to that of a body 2*lbs.* weight moving in the opposite direction, with an uniform velocity of 8*lbs.* per second. Let these moving forces be supposed to act successively, without obstruction, then the 2*lbs.* weight would at the end of one second describe the space of 16 feet, by the force of gravity; and

the other weight, now increased to 2*lbs.*, would at the end of the next second (if not obstructed by gravity) describe 8 feet in the opposite direction. Wherefore, these forces acting simultaneously by means of the fixed pulley, the 2*lbs.* weight will descend 8 feet in 1″, pulling the 1*lb.* weight 8 feet upwards in that time; that is, $16 - 8 = 8$ feet. Hence, when the weights are as 2 to 1, the spaces described by the heavier weight are half what they would be if it descended freely by the force of gravity, without obstruction; that is, as $2 - 1 = 1$, to 2, or the obstruction is $= \frac{1}{2}$.

(2.) If the weights were 3*lbs.* and 1*lb.* the momentum of 1*lb.* would be as before, 16; and if the weight of 1*lb.* were tripled, and its momentum continued the same, viz. 16, its velocity (being in the inverse ratio of this increase of weight) would be $\frac{16}{3} = 5\frac{1}{3}$ feet; for $5\frac{1}{3} \times 3 = 16$. Consequently, the momentum of a body of 1*lb.* weight by the force of gravity, at the end of 1″, is equal to the momentum of a body 3*lbs.* weight, moving with the uniform velocity of $5\frac{1}{3}$ feet per second; and if these weights with these velocities acted simultaneously upon each other by means of the fixed pulley, the 3*lbs.* weight moving with the greater velocity, would descend $16 - 5\frac{1}{3} = 10\frac{2}{3}$ feet in 1″, which is two-thirds of 16 feet. Hence, when the weights are as 3 to 1, the space described by the heavier weight is two-thirds of what it would describe by gravity; that is, as $3 - 1 = 2$, to 3; $3 - 1 : 3$; and the obstruction is $\frac{1}{3}$.

(3.) If the weights were 4 and 1, the momentum of the 1*lb.* weight in 1″ being 16, is equal to the momentum of a weight of 4*lbs.* with a uniform velocity of 4 feet per second; and for the reasons given in art. 1, the heavier weight

would descend $16 - 4 = 12$ feet in $1''$. Hence, when the weights are as 4 to 1, the space described by the heavier body will be to the space that it would describe by gravity as $4 - 1 = 3$, to 4; and the obstruction will be $\frac{1}{4}$, which is half the obstruction when the weights are 2 and 1.

(4.) If the weights were 4*lbs.* and 2*lbs.*, the velocity of the 2*lbs.* weight being 16 feet in $1''$, its resistance is equivalent to that of a 4*lbs.* weight moving uniformly at the rate of 8 feet in one second; and by art. 1, the heavier weight would descend $16 - 8 = 8$ feet in $1''$. Hence the obstruction would be $\frac{2}{4}$ or $\frac{1}{2}$, as in art. 1, where the weights 2 and 1 are in the same ratio as in this art. viz. 4 and 2.

(5.) If the weights are 3 and 2. If the 2*lbs.* weight be increased to 3, that is, half as much again, then its momentum remaining the same, its velocity would be diminished in the same ratio; that is, to two-thirds of $16 = 10\frac{2}{3}$ feet; and in this case, therefore, the space that would be described by the heavier body in $1''$ is $16 - 10\frac{2}{3} = 5\frac{1}{3}$ feet; the obstruction, therefore, in this case, is $\frac{2}{3}$.

(6.) In general the obstruction is as the lesser weight to the greater; and if W represents the greater weight, and w the less; then the obstruction will be expressed by the fraction $\frac{w}{W}$, universally, whatever be the time of descent. This fractional expression $\frac{w}{W}$, infers its ratio to unity, or the number 1; so that it is understood as $\frac{w}{W} : 1 :: $ so is the obstruction of the lesser weight : to the free *descent* by gravity of the greater weight. For it is to be observed, that in our investigations we have ascertained the quantum

of resistance of the lesser weight, by first ascertaining what uniform *velocity* a body, of the same weight with the greater, must possess, in order to make its momentum equal to the momentum of the less weight descending for 1″ by the force of gravity, considering the momentum as the measure of the resistance; and we are enabled to compare the velocity, when found, of this latter increased weight with the velocity of the weight to which it has been made equal, descending by the force of gravity; and this comparison is effected by considering these two equal weights as moving successively one after the other, and then taking the final effect as the same result, which would be accomplished when the forces act simultaneously on those two bodies; and we consider this result the same with that of the less weight against the greater, because it is the result of a resistance equal to that of the less weight against the greater, And by comparing the results, we ascertain that the constant ratio of the obstruction to gravity is as $\frac{w}{W}$ to 1; from which we can ascertain the distance or space described by the heavier weight, by means of the above analogy or proportion. For the space that would be described by gravity alone may be found by formula 2, table, chap. iv.; let S represent that space, equal by the formula to the product of the square of the time into 16 feet; or $S = T^2 \times 16$; consequently, if the weights are given, and S is found, three of the four proportionals are known, from which the fourth, (that is, the space lost by the resistance of the less weight) may be found by the Rule of Three; let s represent this lost space; then $1 : \frac{w}{W} :: S : \frac{wS}{W} = s$; and $S - s$ will be the space through which the heavier body will descend.

(7.) Let the weights be *2lbs.* and *1lb.* the time of descent 1″; then $\frac{1}{2} : 1 : :$ the space lost by the resistance : the space described by gravity; and taking this proportion inversely,

$$1 : \tfrac{1}{2} : : 16 : \frac{16}{2} = 8 \text{ feet} = \text{the space lost by the}$$

obstruction. Consequently, since the space that would be described by the *2lbs.* weight by gravity in 1″ is 16 feet, and the space lost by the resistance of the *1lb.* weight is = 8 feet, ergo $16 - 8 = 8$, is the remaining balance of the space described by gravity above the obstruction-space, and, consequently, is the space described by the weights acting simultaneously, when the balance becomes struck.

(8.) So if the weights are *3lbs.* and *2lbs.* by art. 6 and 7,

$$1 : \tfrac{2}{3} : : 16 : \frac{32}{3} = 10\tfrac{2}{3} \text{ feet};$$

and if the weights are *4lbs.* and *1lb.*,

$$1 : \tfrac{1}{4} : : 16 : \frac{16}{4} = 4lbs.$$

(9.) Instead of taking the time of descent to be 1″, as in the preceding articles, if we take time as 2″; let the weights be 2 and 1, the space that would be described by gravity in 2″ is 64 feet; and by art. 6 and 7,

$$1 : \tfrac{1}{2} : : 64 : \frac{64}{2} = 32 \text{ feet};$$

the space lost; which being subtracted from 64 feet, the space that would be described by the *2lbs.* weight in 2″, by the force of gravity, viz. $64 - 32 = 32$ feet, which is four times the space that the *2lbs.* weight, suspended against the *1lb.* weight, would describe in 1″. So if the weights are *3lbs.* and *2lbs.*,

$$1 : \tfrac{2}{3} :: 64 : \frac{128}{3} = 42\tfrac{2}{3} \text{ feet};$$

which is the space lost in this case; hence, the space that would be described by gravity, acting freely, being 64 feet, and the space lost being $42\frac{1}{3}$ feet; it follows that $64 - 42\frac{2}{3} = 21\frac{1}{3}$. is the space that would be described by the heavier weight, *3lbs.*, in its descent against the counteracting weight of *2lbs.* in 2″, which is four times the space ($5\frac{1}{3}$ feet) that would be described by the *3lbs.* weight against the *2lbs.* weight in 1″. (art. 5.) viz. $5\frac{1}{3} \times 4 = 21\frac{1}{3}$.

(10.) In like manner it will be found, that in whatever ratio the weights may be to each other, the space described by the heavier weight in 2″ will be four times the space described by it in 1″. Likewise, extending the time to 3″, 4″, 5″, &c., the spaces described by the heavier weight will be universally 9, 16, 25, &c. times the space described by the heavier weight in 1″. Thus, taking the time to be 3″, the space described by gravity acting freely, would be 144 feet; hence, by art. 6, taking the weights *2lbs.* and *1lb.*

$$1 : \tfrac{1}{2} :: 144 : \frac{144}{2} = 72 \text{ feet};$$

which is 9 times the space, viz. 8 feet, that the *2lbs.* weight would have descended by gravity in 1″. So taking the time to be 4″, by art. 6.,

$$1 : \tfrac{1}{2} :: 256 : \frac{256}{2} = 128 \text{ feet}; = 8 \times 4^2, \text{\&c.}$$

(11.) Hence, universally, the spaces described by the heavier of any two weights suspended by a fixed pulley against each other, are as the squares of the times; and they, consequently partake of the property of gravitation; differing from it only, because the heavier weight being counteracted can never describe so great a space in the

first second as would be described by gravity acting freely without counteraction.

(12.) From the above reasonings, if acceded to, we might proceed at once to deduce rules for the solution of the different problems which can arise respecting the spaces that would be described by weights, counteracted by other weights suspended against them over a fixed pulley. But another theory is now received, which leads to different results, particularly in small times of descent; which are the more important, because our inquiries in these cases affect only the gravitation of bodies not much above or below the surface of the earth.

(13.) The received theory is this, that the space that will be described by the heavier weight is represented by the fraction $\frac{W - w}{W + w}$; wherefore, if the weights be *2lbs.* and *1lb.*; the space described by the *2lbs.* weight in any time, will be to the space that it would describe by gravity freely in that time, as $\frac{2 - 1}{2 + 1} = \frac{1}{3}$ is to 1. (*Bridge's Mechanics*, ii. 31. art. 2.*) Hence, the obstruction of a weight of *1lb.* against a weight of *2lbs.* would occasion a loss of two-thirds of the space that would be described by gravity, contrary to probability, which would rather lead to the supposition that the obstruction would be in the ratio of the weights, that is, as 1 to 2, or $\frac{1}{2}$, not $\frac{2}{3}$. So by the received theory, if the weights are *3lbs.* and *1lb.*; then in 1″, $1 : \frac{3 - 1}{3 + 1}$; as

* *Whewell's Mechanics*, p. 289, art. 226. In this case the space that would be described by the *2lbs.* weight would be found as follows :

$$1 : \tfrac{1}{3} :: 16 : \frac{16}{3} = 5\tfrac{1}{3} \text{ feet.}$$

the space that would be described by the *3lbs.* weight by gravity not obstructed : is to the space that would be described by the *3lbs.* weight obstructed by the *1lb.* weight; that is, $1 : \frac{2}{4} :: 16 : \frac{32}{4} = 8$ feet; hence, according to the received opinion *one-third* part of the greater weight suspended against it would (contrary to probability and expectation,) diminish the space of the triple weight one half. Also, if the weights were *4lb.* and *1lb.*; the space described by the heavier body in its descent would be $\frac{4-1}{4+1} : \frac{3}{5} \times 16; = \frac{4}{5} = 9\frac{3}{5}$ feet in 1″. But the effect of *4lbs.* against *1lb.* is the sum of the effects of the parts composing the whole *4lbs.* weight; that is, it is the sum of the effects of two weights of *2lbs.* each; but these effects are (by this theory) $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$; whereas, by this theory also, the effect of a weight of *4lbs.* against a weight of *1lb.* is $\frac{3}{5}$ not $\frac{2}{3}$; that is, by reducing these fractions to a common denominator $\frac{9}{15}$ths, instead of $\frac{10}{15}$ths; which it should be to make this theory consistent with itself. Again, by this theory, if the weights were *4lbs.* and *2lbs.*, the *4lbs.* weight would descend with a velocity of $\frac{4-2}{4+2} \times 16, = \frac{2}{6} \times 16,$ or $5\frac{1}{3}$ feet per second; but if the weights were *4lbs.* and *3lbs.*, the *4lbs.* weight would descend with a velocity of $\frac{4-3}{4+3} \times 16 = \frac{1}{7} \times 16 = 2\frac{2}{7}$ feet per second; so that, by adding 50 per cent. to the obstruction, the obstruction is more than doubled. The truth is, that the received theory is not founded on correct reasoning, but upon dogmas mistaken for reasoning. The comparison is not (as it should be) between the momenta of the two weights, between which there is a constant ratio; but is between the velocity of gravity, and the obstructed velocity of the heavier weight; between which two compared velocities there does not subsist any constant ratio whatever.

(14.) Mr. Bridge supports this received theory (vol. ii. art. 1. p. 30.) as follows: "The force which generates *momentum* in a body is called the *moving force;* and the force which generates *velocity* is called the *accelerative force;* if, therefore, for the momentum and velocity of a body, we substitute the *causes which produce them*, it may be said that the *accelerative force* is as the *moving force* directly, and the *quantity of matter moved* inversely," &c. In answer to this reasoning, which he makes the foundation of the received theory, it is enough to say, that no force can generate momentum without generating velocity, nor velocity without momentum; that moving force generates both momentum and velocity; and that there is no foundation for dividing the force which produces both, into two forces, one producing momentum and the other velocity. He does not here use the expression accelerating force as meaning a force generating *increments* of velocity, but as the force which generates an uniform velocity.

(15.) It is not stated by Mr. Bridge (*Mechanics*, part iii. p. 33), that the results of this theory are confirmed by experiments; but it would be inferred that they are, inasmuch as he refers to a machine invented by Mr. Atwood for ascertaining the rates of descent of weights suspended over a fixed pulley against lighter weights; which experiments are stated no further than to this extent, viz. that whatever space the heavier weight describes in the first second in its descent, it will describe 4 times the space in the second second, 9 times in the third, 16 times in the fourth, &c. conformably to the properties of unobstructed gravitation. It would not have been a difficult experiment to suspend first 2*lbs.* against 1*lb.*, and afterwards 4*lbs.* against 1*lb.*; and a theory established upon, that is, consistent with those and other experiments, would have been entitled to great weight, even if opposed to

abstract demonstration; but no such experiments are brought forward by Mr. Bridge in support of his theory; (for the present author conceives that Mr. Bridge originated it. See the third page of the advertisement to *Bridge's Mechanics*).

(16.) It is to be observed that these experiments are confirmatory of our theory; for a greater weight against a less, will gravitate 4 times as far in 2″ as in 1″, 9 times as far in 3 as in 1, &c. as appears in art. 11; that is, the spaces described by the heavier body are as the squares of the times; hence the formulæ for gravitation will answer in these cases if g represents the space described by the greater weight in 1″.

Problem 1.

(17.) To find the space which the greater of two given weights suspended against each other over a fixed pulley would describe in any given time.

$S - s$ is the space required to be found (art. 6.)

$S = 16 \times T^2$, and may therefore be found (art. 6.)

And $1 : \frac{w}{W} :: S : s$; that is, $s = \frac{wS}{W}$ (art. 6.)

And because w, S, and W, are given, s may be found by the rules of arithmetic; that is, the question becomes arithmetically stated. (*New Introduction*, part ii. p. 200; the technical method.)

EXAMPLE *.

A weight of 4*lbs.* is attached to a weight of 1*lb.* by means of a cord over a fixed pulley. How far will the 4*lbs.* weight descend in 3 seconds.

* The first two of the following Examples are the same as in *Bridge's Mechanics*, part iii. pp. 34, 5; so that the different results of the two theories may be compared with each other.

Here W is given $= 4$, and w is given $= 1$, and S may be found $= T^2 \times 16 = 144$.

And $S = \frac{wS}{W} = \frac{1 \times 144}{4} = 36$; the space lost by the resistance.

And $S - s$, that is, $144 - 36 = 108$ feet; the space through which the 4*lbs.* weight would descend. [By Mr. Bridge's theory the 4*lbs.* weight would descend 86·76 feet in 3″.]

Problem 2.

(18.) To find the time in which the greater of any two given weights will descend through a given space.

Here W and w are given, and $S - s$ is given, and T is sought. But because the time is not given, S cannot be found by the general formulæ for gravitation; but if we consider S in this case as the space that would be described by W by gravity in 1″, then $S = 16$ feet; from which we can find the value of s, or the space lost by the resistance in 1″, $= \frac{wS}{W}$; subtracting which from 16 feet, we ascertain the space described by W (as obstructed by w) in the first second. And because the space described in the time which is sought (that is $S - s$) is given, and by formula 5, table, chap. iv. $T = \left(\frac{s}{g}\right)^{\frac{1}{2}} \therefore T = \left(\frac{S-s}{g}\right)^{\frac{1}{2}}$ in which expression g, the denominator, represents the space described by W in the first second.

EXAMPLE.

Two weights, one of 17 oz., and the other 16 oz., are suspended freely over a fixed pulley: in what time will the greater weight descend through 12 feet?

Put S as representing the given quantity $S - s = 12$.

Here $T = \left(\frac{S}{g}\right)^{\frac{1}{2}}$, of which quantities S is given, and T sought; and if g can be found, T can be found.

Here (by art. 16.) g represents the space described by W in the first second; and because

$$1 : \frac{w}{W} : : 16 : \text{the space lost in } 1''.$$

$$1 : \frac{16}{17} : : 16 : \frac{256}{17} = 15 \text{ very nearly} = \text{space lost in } 1''.$$

And subtracting this lost space from the space that would be described by gravity in $1''$, $16 - 15 = 1 = g$, the space described by W in $1''$. Wherefore, in this case,

$$T = \left(\frac{12}{1}\right)^{\frac{1}{2}} = \sqrt{12} = 3{\cdot}46 \text{ seconds, nearly.}$$

[Mr. Bridge's theory makes $T = 4{\cdot}96''$; for as his theory makes the spaces descended *less* than ours, it occasions the time in which the given space is described to be *greater* than ours.]

Problem 3.

(19.) To find the velocity which the greater of any two given weights suspended over a fixed pulley will acquire in descending through a given space.

By formula 6, table, chap. iv. $V = 2 \times (gS)^{\frac{1}{2}}$.

Here S is given, and if g, that is, the space described by W in the first second, can be found, V is given.

EXAMPLE.

Let the weights be 17 oz., and 16 oz., as in the last example, and the space described by W, be 12 feet; and let it be required to find the velocity acquired by W at the end of that descent.

By the solution of the last example $g = 1$; wherefore, $V = 2 \times \sqrt{12} = 2 \times 3{\cdot}46 = 6{\cdot}92$ feet per second (which Mr. Bridge makes 4·83 feet per second; because as by his theory, less spaces are described, less velocities will be acquired).

THE END.

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